Spectral characterization of new classes of multicone graphs

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Abstract. This paper deals with graphs that are known as multicone graphs. A multicone graph is a graph obtained from the join of a clique and a regular graph. Let \( w, l, m \) be natural numbers and \( k \) is a natural number. It is proved that any connected graph cospectral with multicone graph \( K_w \bowtie mECP_l^k \) is determined by its adjacency spectra as well as its Laplacian spectra, where \( ECP_l^k = K_3^k, 3^k, \ldots, 3^k \). Also, we show that complements of some of these multicone graphs are determined by their adjacency spectra. Moreover, we prove that any connected graph cospectral with these multicone graphs must be perfect. Finally, we pose two problems for further researches.

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1. Introduction

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [4, 5, 10, 12, 19]. Let \( \Gamma \) be a graph with \( n \) vertices, \( V(\Gamma) \) and \( E(\Gamma) \) be the sets of vertices and edges of \( \Gamma \), respectively. The complement of a graph \( \Gamma \), denoted by \( \overline{\Gamma} \), is the graph on the vertices set of \( \Gamma \) such that two vertices of \( \overline{\Gamma} \) are adjacent if and only if they are not adjacent in \( \Gamma \). The union of (disjoint) graphs \( \Gamma_1 \) and \( \Gamma_2 \) is denoted by \( \Gamma_1 \cup \Gamma_2 \), is the graph whose vertices (respectively, edges) set is the union of vertices (respectively, edges) set of \( \Gamma_1 \) and \( \Gamma_2 \). A graph consisting of \( k \) disjoint copies of an arbitrary graph \( \Gamma \) will be denoted by \( k\Gamma \). The join of two vertex disjoint graphs \( \Gamma_1 \) and \( \Gamma_2 \) is the graph obtained from \( \Gamma_1 \cup \Gamma_2 \) by joining each vertex in \( \Gamma_1 \) with every vertex in \( \Gamma_2 \). It is denoted by \( \Gamma_1 \bowtie \Gamma_2 \). Let \( \Gamma \) be a graph with adjacency matrix \( A(\Gamma) \). The characteristic polynomial of \( \Gamma \) is \( \det(\lambda I - A(\Gamma)) \), and denoted by \( P_\Gamma(\lambda) \). The roots of \( P_\Gamma(\lambda) \) are called the adjacency eigenvalues of \( A(\Gamma) \). The eigenvalues and the spectrum of \( A(\Gamma) \) are also called the eigenvalues and the
If we consider a matrix $L = D - A$ instead of $A$, where $D$ is the diagonal matrix of degree of vertices (in $\Gamma$), we get the Laplacian eigenvalues and the Laplacian spectrum, while in the case of matrix $SL(G) = D(\Gamma) + A(\Gamma)$, we get the signless Laplacian eigenvalues and the signless Laplacian spectrum, respectively. Since both matrices $A(\Gamma)$ and $L(\Gamma)$ are real symmetric matrices, their eigenvalues are all real numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of $\Gamma$ with multiplicities $m_1, m_2, \ldots, m_s$, respectively. We denote the adjacency spectrum of $\Gamma$ by $Spec(\Gamma) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_s]^{m_s}\}$. Two graphs $\Gamma$ and $\Lambda$ are called cospectral, if $Spec(\Gamma) = Spec(\Lambda)$. A graph $\Gamma$ is said to be determined by its spectrum or DS for short, if $Spec(\Gamma) = Spec(\Lambda)$, follows that $\Gamma \cong \Lambda$. About the background of the question "which graphs are determined by their spectrums?", we refer to [15]. The friendship graph $F_n$ consists of $n$ edge-disjoint triangles that all of them meeting in one vertex, where $n$ is a natural number (see Figure 1). The friendship (or Dutch windmill or n-fan) graph $F_n$ is the graph that can be constructed by coalescing $n$ copies of the cycle graph $C_3$ of length 3 with a common vertex. By construction, the friendship graph $F_n$ is isomorphic to the windmill graph $Wd(3,n)$ [11]. The friendship theorem of Paul Erdős, Alfred Réyni and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. In [17, 18], it has been proposed that the friendship graph is DS with respect to its adjacency spectrum. This conjecture studied in [2, 8]. It is claimed in [8] that conjecture is valid. In [7], it is proved that if $\Gamma$ is any graph cospectral with $F_n$ $(n \neq 16)$, then $\Gamma \cong F_n$. Abdollahi and Janbaz [3] presented a proof in special case of this topic. They proved that any connected graph cospectral with $F_n$ is isomorphic to $F_n$. Abdian and Miražal [1] characterized new classes of multicone graphs. In this paper, we present new classes of multicone graphs that friendship graphs are special classes of them and we show these graphs are DS with respect to their spectra. The plan of the present paper is as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any connected graph cospectral with multicone graph $K_w \triangle mECP_l^k$ (see Figures 1 and 2, for example) must be regular or bidegreed (Lemma 3.2). In Subsection 3.2, we prove that any connected graphs cospectral with $K_w \triangle mECP_l^k$ is determined by its adjacency spectra (Theorem 3.4). In Subsection 3.3, we prove that complement of $K_w \triangle mECP_l^k$ is DS with respect to their adjacency spectra (Theorem 3.7). In Subsection 3.4, we show that graphs $K_w \triangle mECP_l^k$ are DS with respect to their Laplacian spectra (Theorem 3.8). In Subsection 3.5, we show that any connected graph cospectral with multicone graph $K_w \triangle mECP_l^k$ must be perfect. We conclude with final remarks and open problems in Section 4.

2. Preliminaries

In this section, we give some facts that will be used in the proof of the main results.

A walk of length $m$ in a graph $\Gamma(V, E)$ is an alternating sequence:

$v_1l_1v_2l_2v_3l_3v_m l_{m+1}$