Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup $S$. In this paper, we prove that the mapping $T_\mu$ introduced by a mean on a subspace of $B(S)$, has many properties of the mappings in the representation $S$, in Banach and locally convex spaces.

**Keywords**: Representation; nonexpansive; attractive point; directed graph; mean.

1. **Introduction**

Suppose that $C$ is a nonempty closed, convex subset of a reflexive Banach space $E$, $S$ a semigroup, $\mathcal{S} = \{T_s : s \in S\}$ a representation of $S$ as self mappings on $C$ such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$ and $X$ be a subspace of $B(S)$ such that the mapping $t \to \langle T(t)x, x^* \rangle$ be an element of $X$ for each $x \in C$ and $1 \in X$ and $x^* \in E$, and $\mu$ be a mean on $X$. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$. The relations between the representation $\mathcal{S}$ and the mapping $T_\mu$ have been interesting for many years. For example we can see [7, 8, 11, 12].

In this paper, we study some relations between the representation $\mathcal{S}$ and $T_\mu$ in Banach and locally convex spaces.
The space of all bounded real-valued functions defined on $S$ with supremum norm is denoted by $l^\infty(S)$. $l_s$ and $r_s$ in $l^\infty(S)$ are defined as follows: $(l_t g)(s) = g(ts)$ and $(r_t g)(s) = g(st)$, for all $s \in S$, $t \in S$ and $g \in l^\infty(S)$.

Suppose that $X$ is a subspace of $l^\infty(S)$ containing 1 and let $X^*$ be its topological dual space. An element $m$ of $X^*$ is said to be a mean on $X$, provided $\|m\| = m(1) = 1$. For $m \in X^*$ and $g \in X$, $m_t(g(t))$ is often written instead of $m(g)$. Suppose that $X$ is left invariant (respectively, right invariant), i.e., $l_t(X) \subset X$ (respectively, $r_t(X) \subset X$) for each $s \in S$. A mean $m$ on $X$ is called left invariant (respectively, right invariant), provided $m(l_t g) = m(g)$ (respectively, $m(r_t g) = m(g)$) for each $t \in S$ and $g \in X$. $X$ is called left (respectively, right) amenable if $X$ possesses a left (respectively, right) invariant mean. $X$ is amenable, provided $X$ is both left and right amenable.

Let $D$ be a directed set in $X$ and let $\{m_\alpha : \alpha \in D\}$ \cite{1} §1.1, p. 5. A net $\{m_\alpha : \alpha \in D\}$ of means on $X$ is called left regular, provided

$$\lim_{\alpha \in D} \|l_t^* m_\alpha - m_\alpha\| = 0,$$

for every $t \in S$, where $l_t^*$ is the adjoint operator of $l_t$.

Let $E$ be a reflexive Banach space. Let $g$ be a function on $S$ into $E$ such that the weak closure of $\{g(s) : s \in S\}$ is weakly compact and suppose that $X$ is a subspace of $l^\infty(S)$ owning all the functions $s \rightarrow \langle g(s), x^* \rangle$ with $x^* \in E^*$. We know from \cite{4} that, for any $m \in X^*$, there exists a unique element $g_m$ in $E$ such that $\langle g_m, x^* \rangle = m_x \langle f(s), x^* \rangle$ for all $x^* \in E^*$. We denote such $g_m$ by $\int g(s)m(s)$. Moreover, if $m$ is a mean on $X$, then from \cite{6},
\[ \int g(s)m(s) \in \text{co} \{g(s) : s \in S\} \], where \( \text{co} \{g(s) : s \in S\} \) denotes the closure of the convex hull of \( \{g(s) : s \in S\} \).

Recall the following definitions:

(1) suppose that \( S \) be semigroup. Let \( C \) be a nonempty closed and convex subset of \( E \). Then, a family \( S = \{T_s : s \in S\} \) of mappings from \( C \) into itself is called a representation of \( S \) as nonexpansive mappings on \( C \) into itself if \( S \) satisfies the following:

(1) \( T_s x = T_s T_t x \) for all \( s, t \in S \) and \( x \in C \);
(2) for every \( s \in S \) the mapping \( T_s : C \to C \) is nonexpansive.

We denote by \( \text{Fix}(S) \) the set of common fixed points of \( S \), that is \( \text{Fix}(S) = \bigcap_{s \in S} \{x \in C : T_s x = x\} \).

(2) Let \( E \) be a real Banach space and \( C \) be a subset of \( E \). We denote by \( \text{Fix}(T) \) the set of fixed points of a mapping \( T : C \to C \). In this note, a mapping \( T : C \to C \) is called:

(a) nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in C \);
(b) quasi nonexpansive \(^{[11]} \) if \( \|Tx - f\| \leq \|x - f\| \) for all \( x \in C \) and \( f \in \text{Fix}(T) \);
(c) strongly quasi nonexpansive \(^{[11]} \) if \( \|Tx - f\| \leq \|x - f\| \) for all \( x \in C \setminus \text{Fix}(T) \) and \( f \in \text{Fix}(T) \);
(d) \( F \)-quasi nonexpansive (for a subset \( F \subseteq \text{Fix}(T) \)) if \( \|Tx - f\| \leq \|x - f\| \) for all \( x \in C \) and \( f \in \text{Fix}(T) \);
(e) strongly \( F \)-quasi nonexpansive \(^{[11]} \) (for a subset \( F \subseteq \text{Fix}(T) \)) if \( \|Tx - f\| \leq \|x - f\| \) for all \( x \in C \setminus \text{Fix}(T) \) and \( f \in \text{Fix}(T) \), and
(f) retraction \(^{[11]} \) if \( T^2 = T \).

(3) Lau and Zhang \(^{[7]} \), extend asymptotically nonexpansive definition as follows:
let $E$ be a Banach space and $C \subset E$. A mapping $T : C \to C$ is called asymptotically nonexpansive provided for all $x, y \in C$ the following inequality holds:

1. \[
\limsup_{n \to \infty} \|T^n x - T^n y\| \leq \|x - y\|
\]

(The notion of asymptotically nonexpansive mappings was first introduced by Goebel and Kirk in 1972),

4. Suppose that $S = \{T_s : s \in S\}$ is a representation of a semigroup $S$ on a set $C$ in a Banach space $E$. An element $a \in E$ is called an asymptotically attractive point of $S$ for $C$ provided

2. \[
\limsup_{n \to \infty} \|a - T^n.t.x\| \leq \|a - x\|
\]

for all $t \in S$ and $x \in C$.

5. Suppose that $S = \{T_s : s \in S\}$ is a representation of a semigroup $S$ on a set $C$ in a Banach space $E$. $S$ is called an asymptotically representation of $S$ provided

3. \[
\limsup_{n \to \infty} \|T_i^n x - T_i^n y\| \leq \|x - y\|
\]

for all $t \in S$ and $x, y \in C$.

6. Suppose that $Q$ is a family of seminorms on a locally convex space $X$ which determines the topology of $X$ and $C$ be a nonempty closed and convex subset of $X$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ (to see more details refer to [5]). A mapping $T$ of $C$ into itself is called $Q$-nonexpansive if $q(Tx - Ty) \leq q(x - y)$, whenever $(x, y) \in E(G)$ for any $x, y \in C$ and $q \in Q$, and a mapping $f$ is a $Q$-contraction on $E$ if $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in E$ such that $0 \leq \beta < 1.$
(7) Suppose that $Q$ is a family of seminorms on a locally convex space $X$ which determines the topology of $X$. The locally convex topology $\tau_Q$ is separated if and only if the family of seminorms $Q$ possesses the following property: for each $x \in X \setminus \{0\}$ there exists $q \in Q$ such that $q(x) \neq 0$ or equivalently 
\[ \bigcap_{q \in Q} \{ x \in X : q(x) = 0 \} = \{0\} \ (\text{see [2]}). \]

The following Lemma which we will use, is well known.

**Lemma 2.1.** [13, 4] Suppose that $g$ is a function of $S$ into $E$ such that the weak closure of $\{g(t) : t \in S\}$ is weakly compact and let $X$ be a subspace of $B(S)$ containing all the functions $t \to \langle g(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element $g_\mu$ in $E$ such that
\[ \langle g_\mu, x^* \rangle = \mu_t \langle g(t), x^* \rangle \]
for all $x^* \in E^*$. Moreover, if $\mu$ is a mean on $X$ then
\[ \int g(t) \, d\mu(t) \in \overline{co} \{ g(t) : t \in S \}. \]

We can write $g_\mu$ by
\[ \int g(t) \, d\mu(t). \]

3. **Some results of Hahn Banach theorem**

Suppose that $Q$ is a family of seminorms on a locally convex space $X$ which determines the topology of $X$ and $q \in Q$ is a seminorm. Let $Y$ be a subset of $X$, we put $q_Y^*(f) = \sup \{|f(y)| : y \in Y, q(y) \leq 1\}$ and $q^*(f) = \sup \{|f(x)| : x \in X, q(x) \leq 1\}$, for every linear functional $f$ on $X$. Observe that, for each $x \in X$ that $q(x) \neq 0$ and $f \in X^*$, then $|\langle x, f \rangle| \leq q(x)q^*(f)$. We will make use of the following Theorems.
**Theorem 3.1.** Suppose that \( Q \) is a family of seminorms on a real locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \) is a continuous seminorm and \( Y \) is a vector subspace of \( X \) such that \( Y \cap \{ x \in X : q(x) = 0 \} = \{ 0 \} \). Let \( f \) be a real linear functional on \( Y \) such that \( q_Y^*(f) < \infty \). Then there exists a continuous linear functional \( h \) on \( X \) that extends \( f \) such that \( q_Y^*(f) = q^*(h) \).

**Proof.** If we define \( p : X \to \mathbb{R} \) by \( p(x) = q_Y^*(f)q(x) \) for each \( x \in X \), then we have \( p \) is a seminorm on \( X \) such that \( f(x) \leq p(x) \), for each \( x \in Y \). Because, if \( x = 0 \), clearly \( f(x) = 0 \) and \( 0 \leq p(x) \). On the other hand, if \( x \in Y \) and \( x \neq 0 \) then from our assumption, \( q(x) \neq 0 \) and \( q(x) = 1 \). Therefore, we have \( f(\frac{x}{q(x)}) \leq q_Y^*(f) \), then \( f(x) \leq q_Y^*(f)q(x) = p(x) \). Since \( q \) is continuous, \( p \) is also a continuous seminorm, therefore by the Hahn-Banach theorem (Theorem 3.9 in [10]), there exists a linear continuous extension \( h \) of \( f \) to \( X \) that \( h(x) \leq p(x) \) for each \( x \in X \). Hence, since \( X \) is a vector space, we have

\[(4) \quad |h(x)| \leq q_Y^*(f)q(x), (x \in X)\]

and hence, \( q^*(h) \leq q_Y^*(f) \). Moreover, since \( q_Y^*(f) = \sup\{|f(x)| : x \in Y, q(x) \leq 1\} \leq \sup\{|h(x)| : q(x) \leq 1\} = q^*(h) \), we have \( q_Y^*(f) = q^*(h) \). \( \square \)

**Theorem 3.2.** Suppose that \( Q \) is a family of seminorms on a real locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \) a nonzero continuous seminorm. Let \( x_0 \) be a point in \( X \). Then there exists a continuous linear functional on \( X \) such that \( q_Y^*(f) = 1 \) and \( f(x_0) = q(x_0) \).

**Proof.** Let \( Y := \{ y \in X : q(y) = 0 \} \). We consider two cases:

Case 1. Let \( x_0 \in Y \). Since \( q \) is continuous, \( Y \) is a closed subset of \( X \). Indeed, if \( x \in \overline{Y} \) and \( x_\alpha \in Y \) is a net such that \( x_\alpha \to x \). Then we have \( q(x) = \lim q(x_\alpha) = 0 \), hence \( x \in Y \), then \( Y \) is a closed. Let \( y_0 \) be a point in
There exists some $r > 0$ such that $q(y - y_0) > r$ for all $y \in Y$. Suppose that $Z = \{y + \alpha y_0 : \alpha \in \mathbb{R}, y \in Y\}$, the vector subspace generated by $Y$ and $y_0$. Then we define $h : Z \to \mathbb{R}$ by $h(y + \alpha y_0) = \alpha$. Obviously, $h$ is linear and we have also $r|h(y + \alpha y_0)| = r|\alpha| < |\alpha|q(\alpha^{-1}y + y_0) = q(y + \alpha y_0)$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Therefore $h$ is a linear functional on $Z$ that $q^*_Z(h)$ dose not exceed $r^{-1}$. Putting $p = r^{-1}q$, we have $p$ is a continuous seminorm such that $h(z) \leq p(z)$ for each $z \in Z$, therefore by the Hahn-Banach theorem (Theorem 3.9 in [10]), there exists a linear continuous extension $L$ of $h$ to $X$ that $L(x) \leq p(x)$ for each $x \in X$. We have also $L(x_0) = h(x_0) = q(x_0) = 0$. Now, since $q^*_Z(h) \neq 0$, we have also $q^*(L) \neq 0$, we can define $f := \frac{L}{q^*(L)}$. Hence, $f$ is a linear continuous functional on $Z$ that $f(x_0) = q(x_0) = 0$ and also $q^*(f) = 1$.

Case 2. Let $x_0 \notin Y$. Let $Z := \{\alpha x_0 : \alpha \in \mathbb{R}\}$ that is the vector subspace generated by $x_0$. If we define $h(\alpha x_0) = \alpha q(x_0)$ then $h$ is a linear functional on $Z$ that $h(x_0) = q(x_0)$ and also $q^*_Z(h) = 1$. Since $Z \cap Y = \{0\}$, from Theorem 3.1, there exists a continuous linear extension $f$ of $h$ to $X$ such that $q^*(f) = q^*_Z(h) = 1$. Obviously, $f(x_0) = q(x_0)$.

\[ \square \]

4. Main result

In the following theorem, we prove that $T_\mu$ inherits some properties of representation $S$ in Banach spaces.

**Theorem 4.1.** Suppose that $C$ is a nonempty closed, convex subset of a reflexive Banach space $E$, $S$ a semigroup, $S = \{T_s : s \in S\}$ a representation of $S$ as self mappings on $C$ such that weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$ and $X$ be a subspace of $B(S)$ such that the mapping $t \to \langle T(t)x, x^* \rangle$
be an element of $X$ for each $x \in C$ and $1 \in X$ and $x^* \in E$, and $\mu$ be a mean on $X$. If we write $T_\mu x$ instead of $\int T_t x \, d\mu(t)$, then the following hold.

(a) If $S = \{T_s : s \in S\}$ be a representation of $S$ as asymptotically nonexpansive self mappings on $C$, then $T_\mu$ is an asymptotically nonexpansive self mapping on $C$,
(b) $T_\mu x = x$ for each $x \in \text{Fix}(S)$,
(c) $T_\mu x \in \overline{\text{co}} \{T_t x : t \in S\}$ for each $x \in C$,
(d) if $X$ is $r_s$-invariant for each $s \in S$ and $\mu$ is right invariant, then $T_\mu T_t = T_\mu$ for each $t \in S$,
(e) if $a \in X$ is an asymptotically attractive point of $S$, then $a$ is an asymptotically attractive point of $T_\mu$,
(f) let $S = \{T_s : s \in S\}$ be a representation of $S$ as affine self mappings on $C$, then $T_\mu$ is an affine self mapping on $C$,
(g) let $P$ be a self mappings on $C$ that commutes with $T_s \in S = \{T_s : s \in S\}$ for each $s \in S$ then $T_\mu$ commutes with $P$,
(h) let $S = \{T_s : s \in S\}$ be a representation of $S$ as quasi nonexpansive self mappings on $C$, then $T_\mu$ is a $\text{Fix}(S)$-quasi nonexpansive self mapping on $C$,
(i) let $S = \{T_s : s \in S\}$ be a representation of $S$ as $F$-quasi nonexpansive self mappings on $C$ (for a subset $F \subseteq \text{Fix}(S)$), then $T_\mu$ is an $F$-quasi nonexpansive self mapping on $C$,
(j) let $S = \{T_s : s \in S\}$ be a representation of $S$ as strongly $F$-quasi nonexpansive self mappings on $C$ (for a subset $F \subseteq \text{Fix}(S)$), then $T_\mu$ is an strongly $F$-quasi nonexpansive self mapping on $C$,
(k) let $S = \{T_s : s \in S\}$ be a representation of $S$ as retraction self mappings on $C$, then $T_\mu$ is a retraction self mapping on $C$,
(l) let $E = H$ be a Hilbert space and $S = \{T_s : s \in S\}$ be a representation of $S$ as monotone self mappings on $H$, then $T_\mu$ is a monotone self mapping on $H$.

Proof. (a) Since $S$ is a representation as asymptotically nonexpansive self mappings on $C$, from part (b) of Theorem 3. 1 7 in [9] there exists an integer $m_0 \in \mathbb{N}$ such that

$$\sup_{t \in S} \|T^n_t x - T^n_t y\| \leq \|x - y\|$$

for all $n \geq m_0$, $x, y \in C$. Suppose that $x^*_1 \in J(T^n_\mu x - T^n_\mu y)$ and $x, y \in C$. We know from [4] that, for any $\mu \in X^*$, there exists a unique element $f_\mu$ in $E$ such that

$$\langle f_\mu, x^*_1 \rangle = \mu_s \langle f(s), x^*_1 \rangle$$

for all $x^*_1 \in E^*$. Then for all $n \geq m_0$, $x, y \in C$ and $t \in S$ we have

$$\|T^n_\mu x - T^n_\mu y\|^2 = \langle T^n_\mu x - T^n_\mu y, x^*_1 \rangle = \mu_t \langle T^n_t x - T^n_t y, x^*_1 \rangle$$

$$\leq \sup_t \|T^n_t x - T^n_t y\| \|T^n_t x - T^n_t y\|$$

$$\leq \|x - y\| \|T^n_\mu x - T^n_\mu y\|,$$

hence for all $n \geq m_0$, $x, y \in C$ we have

$$\|T^n_\mu x - T^n_\mu y\| \leq \|x - y\|,$$

therefore we have

$$\limsup_{n \to \infty} \|T^n_\mu x - T^n_\mu y\| \leq \|x - y\|.$$

(b) suppose that $x \in \text{Fix}(S)$ and $x^* \in E^*$. Therefore we have

$$\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle$$
(c) this assertion concludes from Lemma 2.1.
(d) to prove this assertion, we have
\[ \langle T_\mu(T_s x), x^* \rangle = \mu t \langle T_s x, x^* \rangle = \mu t \langle T x, x^* \rangle = \langle T_\mu x, x^* \rangle, \]
(e) suppose that \( x_2^* \in J(a - T^n_\mu x), \)
\[ \|a - T^n_\mu x\|^2 = \langle a - T^n_\mu x, x_2^* \rangle = \mu t \langle a - T^n x, x_2^* \rangle \]
\[ \leq \sup_t \|a - T^n x\| \|a - T^n_\mu x\| \]
\[ \leq \|a - x\| \|a - T^n_\mu x\|, \]
hence for all \( n \geq m_0, x \in C \) we have
\[ \|a - T^n_\mu x\| \leq \|a - x\|, \]
therefore we have
\[ \limsup_{n \to \infty} \|a - T^n_\mu x\| \leq \|a - x\|. \]
(f) Suppose that \( x_1^* \in E^*. \) Then for all positive integers \( \alpha, \beta \) such that \( \alpha + \beta = 1, x, y \in C \) and \( t \in S \) we have
\[ \langle T_\mu(\alpha x + \beta y), x_1^* \rangle = \mu t \langle T_t(\alpha x + \beta y), x_1^* \rangle \]
\[ = \mu t \langle \alpha T_t x + \beta T_t y, x_1^* \rangle \]
\[ = \alpha \mu t \langle T_t x, x_1^* \rangle + \beta \mu t \langle T_t y, x_1^* \rangle \]
\[ = \alpha \langle T_\mu x, x_1^* \rangle + \beta \langle T_\mu y, x_1^* \rangle \]
\[ = \langle \alpha T_\mu x + \beta T_\mu y, x_1^* \rangle \]
hence, we have
\[ T_\mu(\alpha x + \beta y) = \alpha T_\mu x + \beta T_\mu y. \]
(g) Let \( P \) commutes with \( T_s \in \mathcal{S} = \{T_s : s \in S\} \) for each \( s \in S \) and \( x^*_1 \in E^* \). Then from (5), for each \( x \in C \) and \( t \in S \) we have

\[
\langle T_\mu Px, x^*_1 \rangle = \mu_t \langle T_t Px, x^*_1 \rangle \\
= \mu_t \langle PT_t x, x^*_1 \rangle \\
= \langle PT_\mu x, x^*_1 \rangle,
\]

then \( T_\mu P = PT_\mu \).

(h) Let \( \mathcal{S} = \{T_s : s \in S\} \) be a representation of \( S \) as quasi nonexpansive self mappings on \( C \), then for each \( t \in S \) we have \( \|T_t x - f\| \leq \|x - f\| \) for each \( f \in \text{Fix}(T_t) \) and \( x \in C \). Suppose that \( f \in \text{Fix}(\mathcal{S}) \) and \( x^*_2 \in J(T_\mu x - f) \), then from (5), we have

\[
\|T_\mu x - f\|^2 = \langle T_\mu x - f, x^*_2 \rangle = \mu_t \langle T_t x - f, x^*_2 \rangle \\
\leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \\
\leq \|x - f\| \|T_\mu x - f\|,
\]

then we have

\[
\|T_\mu x - f\| \leq \|x - f\|,
\]

then \( T_\mu \) is a \( \text{Fix}(\mathcal{S}) \)-quasi nonexpansive self mapping on \( C \).

(i) Let \( \mathcal{S} = \{T_s : s \in S\} \) be a representation of \( S \) as \( F \)-quasi nonexpansive self mappings on \( C \) that \( F \subseteq \text{Fix}(\mathcal{S}) \), then for each \( t \in S \) we have \( \|T_t x - f\| \leq \|x - f\| \) for each \( f \in F \) and \( x \in C \). Suppose that \( f \in F \), \( x \in C \) and
\[ x_2^* \in J(T_\mu x - f), \text{ then from (5), we have} \]
\[ \|T_\mu x - f\|^2 = \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \]
\[ \leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \]
\[ \leq \|x - f\| \|T_\mu x - f\|, \]
then we have
\[ \|T_\mu x - f\| \leq \|x - f\|, \]
then \( T_\mu \) is an \( F \)-quasi nonexpansive self mapping on \( C \).

(j) Let \( S = \{T_s : s \in S\} \) be a representation of \( S \) as strongly \( F \)-quasi nonexpansive self mappings on \( C \) such that \( F \subseteq \text{Fix}(S) \), then for each \( t \in S \) we have \( \|T_t x - f\| < \|x - f\| \) for each \( x \in C \setminus F \) and \( f \in F \). Suppose that \( f \in F, x \in C \setminus F \) and \( x_2^* \in J(T_\mu x - f) \), then from (5), we have
\[ \|T_\mu x - f\|^2 = \langle T_\mu x - f, x_2^* \rangle = \mu_t \langle T_t x - f, x_2^* \rangle \]
\[ \leq \sup_t \|T_t x - f\| \|T_\mu x - f\| \]
\[ < \|x - f\| \|T_\mu x - f\|, \]
then we have
\[ \|T_\mu x - f\| < \|x - f\|, \]
then \( T_\mu \) is a strongly \( F \)-quasi nonexpansive self mapping on \( C \).

(k) Let \( S = \{T_s : s \in S\} \) be a representation of \( S \) as retraction self mappings on \( C \), then for each \( t \in S \) we have \( T_t^2 = T_t \). Suppose that \( x \in C \)
and \(x_1^* \in E^*\), then from (5), we have
\[
\langle T^2_\mu x, x_1^* \rangle = \mu_t \langle T^2_t x, x_1^* \rangle \\
= \mu_t \langle T_t x, x_1^* \rangle \\
= \langle T_\mu x, x_1^* \rangle,
\]
then \(T^2_\mu = T_\mu\).

(l) Since \(T_s\) is monotone for every \(s \in S\), then we have \(\langle T_s x - T_s y, x - y \rangle \geq 0\) for every \(x, y \in H\) and \(s \in S\). As in the proof of Theorem 1.4.1 in [14] we know that \(\mu\) is positive i.e., \(\langle \mu, f \rangle \geq 0\) for each \(f \in X\) that \(f \geq 0\). Then for each \(x, y \in H\), from (5) we have
\[
\langle T_\mu x - T_\mu y, x - y \rangle = \mu_t \langle T_t x - T_t y, x - y \rangle \geq 0,
\]
then \(T_\mu\) is a monotone self mapping on \(H\).

We will need the following Theorem.

**Theorem 4.2.** Let \(S\) be a semigroup, \(E\) be a real dual locally convex space with real predual locally convex space \(D\) and \(U\) a convex neighbourhood of 0 in \(D\) and \(p_U\) be the associated Minkowski functional. Let \(f : S \to E\) be a function such that \(\langle x, f(t) \rangle \leq 1\) for each \(t \in S\) and \(x \in U\). Let \(X\) be a subspace of \(B(S)\) such that the mapping \(t \to \langle x, f(t) \rangle\) be an element of \(X\), for each \(x \in D\). Then, for any \(\mu \in X^*\), there exists a unique element \(F_\mu \in E\) such that \(\langle x, F_\mu \rangle = \mu_t \langle x, f(t) \rangle\), for all \(x \in D\). Furthermore, if \(1 \in X\) and \(\mu\) is a mean on \(X\), then \(F_\mu\) is contained in \(\text{co}\{f(t) : t \in S\}^{\omega^*}\).
Proof. We define $F_\mu$ by $\langle x, F_\mu \rangle = \mu_t \langle x, f(t) \rangle$ for all $x \in D$. Obviously, $F_\mu$ is linear in $x$. Moreover, from Proposition 3.8 in [10], we have

$$|\langle x, F_\mu \rangle| = |\mu_t \langle x, f(t) \rangle| \leq \sup_t |\langle x, f(t) \rangle| \|\mu\| \leq P_U(x) \|\mu\|,$$

for all $x \in D$. Let $(x_\alpha)$ be a net in $D$ that converges to $x_0$. Then by (6) we have

$$|\langle x_\alpha, F_\mu \rangle - \langle x_0, F_\mu \rangle| = |\langle x_\alpha - x_0, F_\mu \rangle| \leq P_U(x_\alpha - x_0) \|\mu\|,$$

taking limit, since from Theorem 3.7 in [10], $P_U$ is continuous, we have $F_\mu$ is continues on $D$, hence $F_\mu \in E$.

Now, let $1 \in X$ and $\mu$ be a mean on $X$. Then, there exists a net $\{\mu_\alpha\}$ of finite means on $X$ such that $\{\mu_\alpha\}$ converges to $\mu$ with the weak* topology on $X^*$. We may consider that

$$\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \delta_{t_{\alpha,i}}.$$

Therefore,

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle = \langle x, \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \rangle, (\forall x \in D, \forall \alpha),$$

then we have

$$F_{\mu_\alpha} = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \in \text{co}\{f(t) : t \in S\}, (\forall \alpha),$$

now since,

$$\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle \to \mu_t \langle x, f(t) \rangle = \langle x, f(t) \rangle, (x \in D),$$
\( \{F_{\mu_n}\} \) converges to \( F_\mu \) in the weak\(^*\) topology. Hence
\[
F_\mu \in \text{co}\{f(t) : t \in S\}^{\omega^*},
\]
we can write \( F_\mu \) by \( \int f(t) d\mu(t) \).

In the following theorem, we prove that \( T_\mu \) inherits some properties of representation \( S \) in locally convex spaces.

**Theorem 4.3.** Let \( S \) be a semigroup, \( C \) a closed convex subset of a real locally convex space \( E \). Let \( G = (V(G), \mathcal{E}(G)) \) a directed graph such that \( V(G) = C \). Let \( \mathcal{B} \) be a base at 0 for the topology consisting of convex, balanced sets. Let \( Q = \{q_V : V \in \mathcal{B}\} \) which \( q_V \) is the associated Minkowski functional with \( V \). Let \( S = \{T_s : s \in S\} \) be a representation of \( S \) as \( Q\)-\( G \)-nonexpansive mappings from \( C \) into itself and \( X \) be a subspace of \( B(S) \) such that \( 1 \in X \) and \( \mu \) be a mean on \( X \) such that the mapping \( t \rightarrow \langle T_t x, x^* \rangle \) is an element of \( X \) for each \( x \in C \) and \( x^* \in E^* \). If we write \( T_\mu x \) instead of \( \int T_t x d\mu(t) \), then the following hold.

(i) \( T_\mu \) is a \( Q\)-\( G \)-nonexpansive mapping from \( C \) into \( C \),

(ii) \( T_\mu x = x \) for each \( x \in \text{Fix}(S) \),

(iii) If moreover \( E \) is a real dual locally convex space with real predual locally convex space \( D \) and \( C \) a \( w^* \)-closed convex subset of \( E \) and \( U \) a convex neighbourhood of 0 in \( D \) and \( p_U \) is the associated Minkowski functional.

Let the mapping \( t \rightarrow \langle z, T_t x \rangle \) is an element of \( X \) for each \( x \in C \) and \( z \in D \) then \( T_\mu x \in \text{co} \{T_t x : t \in S\}^{\omega^*} \) for each \( x \in C \),

(iv) if \( X \) is \( r_s \)-invariant for each \( s \in S \) and \( \mu \) is right invariant, then \( T_\mu T_t = T_\mu \) for each \( t \in S \),

(v) if \( a \in E \) is an \( Q\)-\( G \)-attractive point of \( S \) then \( a \) is an \( Q\)-\( G \)-attractive point of \( T_\mu \).
Proof. (i) Let \( x, y \in C \) and \( V \in \mathcal{B} \). By Proposition 3.33 in [10], the topology on \( E \) induced by \( Q \) is the original topology on \( E \). By Theorem 3.7 in [10], \( q_V \) is a continuous seminorm and from Theorem 1.36 in [9], \( q_V \) is a nonzero seminorm because if \( x \notin V \) then \( q_V(x) \geq 1 \), hence from Theorem 3.2, there exists a functional \( x^*_V \in X^* \) such that \( q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x^*_V \rangle \) and \( q^*_V(x^*_V) = 1 \), and since from Theorem 3.7 in [10], \( q_V(z) \leq 1 \) for each \( z \in V \), we conclude that \( \langle z, x^*_V \rangle \leq 1 \) for all \( z \in V \). Therefore from Theorem 3.8 in [10], \( \langle z, x^*_V \rangle \leq q_V(z) \) for all \( z \in E \). Hence for each \( t \in S, x, y \in C \) that \( (x, y) \in E(G) \) and \( x^* \in E^* \), from (5), we have

\[
q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x^*_V \rangle = \mu_t \langle T_t x - T_t y, x^*_V \rangle \\
\leq \|\mu\| \sup_t |\langle T_t x - T_t y, x^*_V \rangle| \\
\leq \sup_t q_V(T_t x - T_t y) \\
\leq q_V(x - y),
\]

then we have

\[
q_V(T_\mu x - T_\mu y) \leq q_V(x - y),
\]

for all \( V \in \mathcal{B} \).

(ii) Let \( x \in Fix(S) \) and \( x^* \in E^* \). Then we have

\[
\langle T_\mu x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \mu_t \langle x, x^* \rangle = \langle x, x^* \rangle
\]

(iii) this assertion concludes from Theorem 4.2.

(iv) for this assertion, note that

\[
\langle T_\mu(T_{sx})x^*, x^* \rangle = \mu_t \langle T_{ts}x, x^* \rangle = \mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle
\]
(v) Let $x \in C$ and $V \in \mathcal{B}$. From Theorem 3.2, there exists a functional $x^*_V \in X^*$ such that $q_V(a - T_\mu x) = \langle a - T_\mu x, x^*_V \rangle$ and $q^*_V(x^*_V) = 1$. Since from Theorem 3.7 in [10], $q_V(z) \leq 1$ for each $z \in V$, we conclude that $\langle z, x^*_V \rangle \leq 1$ for all $z \in V$. Therefore from Theorem 3.8 in [10], $\langle z, x^*_V \rangle \leq q_V(z)$ for all $z \in E$. Hence for each $t \in S$, $x, y \in C$ that $(x, y) \in E(G)$ and $x^* \in E^*$, from (5), we have

$$q_V(a - T_\mu x) = \langle a - T_\mu x, x^*_V \rangle = \mu_t \langle a - T_t x, x^*_V \rangle$$

$$\leq \|\mu\| \sup_t |\langle a - T_t x, x^*_V \rangle|$$

$$\leq \sup_t q_V(a - T_t x)$$

$$\leq q_V(a - x),$$

then we have

$$q_V(a - T_\mu x) \leq q_V(a - x),$$

for all $V \in \mathcal{B}$.

\[ \square \]

References


