Fixed Point Properties and $Q$-Nonexpansive Retractions in Locally Convex Spaces

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Abstract. Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. We study the existence of $Q$-nonexpansive retractions for families of $Q$-nonexpansive mappings and prove that a separated and sequentially complete locally convex space $E$ has the weak fixed point property for commuting separable semitopological semigroups of $Q$-nonexpansive mappings. This proves the Bruck’s problem (Pacific J Math 53:59–71, 1974) for locally convex spaces. Moreover, we prove the existence of $Q$-nonexpansive retractions for the right amenable $Q$-nonexpansive semigroups.

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1. Introduction

The first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space was established by Baillon [4]: Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $T$ be a nonexpansive mapping of $C$ into itself. If the set $Fix(T)$ of fixed points of $T$ is nonempty, then for each $x \in C$, the Cesaro means $S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$ converge weakly to some $y \in Fix(T)$. In Baillon’s theorem, putting $y = P x$ for each $x \in C$, $P$ is a nonexpansive retraction of $C$ onto $Fix(T)$ such that $P T^n = T^n P = P$ for all positive integers $n$ and $P x \in \overline{co}\{T^n x : n = 1, 2, \ldots\}$ for each $x \in C$. Takahashi [26] proved the existence of such retractions, ergodic retractions, for non-commutative semigroups of nonexpansive mappings in a Hilbert space: If $S$ is an amenable semigroup, $C$ is a closed, convex subset of a Hilbert space $H$ and $S = \{T_s : s \in S\}$ is a nonexpansive semigroup on $C$ such that $Fix(S) \neq \emptyset$, then there exists a
nonexpansive retraction $P$ from $C$ onto $\text{Fix}(S)$ such that $PT_t = T_t P = P$ for each $t \in S$ and $Px \in \overline{\text{co}} \{T_t x : t \in S \}$ for each $x \in C$. These results were extended to uniformly convex Banach spaces for commutative semigroups in [9] and for amenable semigroups in [12, 13]. For some related results, we refer the readers to the works in [20–23]. In this paper, we find some ergodic retractions for locally convex spaces.

Bruck proved in [6] a Banach space $E$ has the weak fixed point property for commuting semigroups if it has the weak fixed point property. In this research, we prove this problem for locally convex spaces. In 2011 Anakkamatee and S. Dhompongsa [1] obtained this result of Rod for a nonexpansive semigroup on a CAT(0) space in which the $\Delta$-convergence plays the role of weak convergence.

Let $f$ be a function of semigroup $S$ into a reflexive Banach space $E$ such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and let $X$ be a subspace of $B(S)$ containing all the functions $t \rightarrow (f(t), x^*)$ with $x^* \in E^*$. We know from [9] that for any $\mu \in X^*$, there exists a unique element $f_\mu$ in $E$ such that $\langle f_\mu, x^* \rangle = \mu(f(t), x^*)$ for all $x^* \in E^*$. We denote such $f_\mu$ by $\int f(t) \, d\mu(t)$. Moreover, if $\mu$ is a mean on $X$ then from [11], $\int f(t) \, d\mu(t) \in \overline{\text{co}} \{f(t) : t \in S\}$. In this paper, we find such function $f_\mu$ for locally convex spaces.

\section{2. Preliminaries}

The space of all bounded real-valued functions defined on a semigroup $S$ with supremum norm is denoted by $B(S)$, $l_1$ and $r_t$ in $B(S)$ are defined as follows: $(l_t g)(s) = g(ts)$ and $(r_t g)(s) = g(st)$, for all $s \in S$, $t \in S$ and $g \in B(S)$. Suppose that $X$ is a subspace of $B(S)$ containing 1 and let $X^*$ be its topological dual space. An element $m$ of $X^*$ is said to be a mean on $X$, provided $\|m\| = m(1) = 1$. For $m \in X^*$ and $g \in X$, $m_t(g(t))$ is often written instead of $m(g)$. Suppose that $X$ is left invariant (respectively, right invariant), i.e., $l_t(X) \subset X$ (respectively, $r_t(X) \subset X$) for each $s \in S$. A mean $m$ on $X$ is called left invariant (respectively, right invariant), provided $m(l_t g) = m(g)$ (respectively, $m(r_t g) = m(g)$) for each $t \in S$ and $g \in X$. $X$ is called left (respectively, right) amenable if $X$ possesses a left (respectively, right) invariant mean. $X$ is amenable, provided $X$ is both left and right amenable.

Recall the following definitions:

1. A semitopological semigroup is a semigroup $S$ with a Hausdorff topology such that for each $a \in S$ the mappings $s \rightarrow a.s$ and $s \rightarrow s.a$ from $S$ to $S$ are continuous. For example, $(\mathbb{R}, +)$ with the usual topology on $\mathbb{R}$ and any semigroup with the discrete topology are semitopological semigroups,

2. Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$ and $C \subset E$. A mapping $T : C \to C$ is said to be $Q$-nonexpansive provided the following inequality holds:
\[ q(Tx - Ty) \leq q(x - y), \]

for all \( x, y \in C \) and \( q \in Q \),

(3) Let \( C \) be a nonempty closed and convex subset of \( E \). Then, a family \( S = \{ T_s : s \in S \} \) of mappings from \( C \) into itself is said to be a representation of \( S \) as \( Q \)-nonexpansive mapping on \( C \) into itself if \( S \) satisfies the following properties:

(1) \( T_{st}x = T_sT_tx \) for all \( s, t \in S \) and \( x \in C \);

(2) for every \( s \in S \) the mapping \( T_s : C \to C \) is \( Q \)-nonexpansive.

We denote by \( \text{Fix}(S) \) the set of common fixed points of \( S \), that is

\[ \text{Fix}(S) = \bigcap_{s \in S} \{ x \in C : T_sx = x \}, \]

(4) Let \( E \) be a Hausdorff locally convex space. Then \( E \) is quasi-complete if every bounded Cauchy net is convergent. Observe that complete \( \Rightarrow \) quasi-complete \( \Rightarrow \) sequentially complete,

(5) Let \( Q \) be a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \) and suppose \( E \) is Hausdorff and sequentially complete. Consider a sequence \( \{ x_n \} \) and a series \( \sum_{n=1}^{\infty} x_n \) in \( E \). A series is called \( Q \)-absolutely convergent if \( \sum_{n=1}^{\infty} q(x_n) < \infty \) for each \( q \in Q \),

(6) Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \) and \( C \) a closed, convex subset of \( E \). We say that \( E \) has the weakly fixed point property if, for every nonempty weakly compact convex subset \( C \) of \( E \), every \( Q \)-nonexpansive mapping of \( C \) into itself has a fixed point,

(7) Let \( S \) be a semigroup and \( C \) be a closed, convex subset of a locally convex space \( E \). We call a representation \( S = \{ T_s : s \in S \} \) an \( Q \)-nonexpansive representation if

\[ q(T_tx - T_ty) \leq q(x - y), \]

for all \( x, y \in C \) and \( t \in S \).

(8) Let \( C \) be a nonempty subset of a topological space \( X \) and \( D \) a nonempty subset of \( C \). Then a continuous mapping \( P : C \to D \) is said to be a retraction if \( Px = x \) for all \( x \in D \), i.e., \( P^2 = P \). In such case, \( D \) is said to be a retract of \( C \).

(9) Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \). Then \( E \) is separated if and only if \( Q \) possesses the following property:

for every \( x \in X \setminus \{ 0 \} \) there is a seminorm \( q \in Q \) such that \( q(x) \neq 0 \).

(10) Suppose that \( Q \) is a family of seminorms on a locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \). From page 3 in [5], a linear functional \( f : X \to \mathbb{R} \) is continues if there are a constant \( M \geq 0 \) and \( q_1, \ldots, q_n \in Q \) such that \( |f(x)| \leq M \max_{1 \leq i \leq n} q_i(x) \) for all \( x \in X \),

(11) Let \( S \) be a semigroup. Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \) and let \( C \) be
a closed, convex subset of a locally convex space $E$ and $S = \{T_s : s \in S\}$ a $Q$-nonexpansive representation on $C$. A point $a \in E$ is an attractive point of $S$ if $q(a - T_s x) \leq q(a - x)$ for all $x \in C$, $s \in S$ and $q \in Q$.

(12) Let $X$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset $B \subseteq X$ is balanced if for all $x \in B$ and $\alpha$ in the base field such that $|\alpha| \leq 1$, we have $\alpha x \in B$.

3. Some Applications of Hahn Banach Theorem in Locally Convex Spaces

Let $Y$ be a subset of a locally convex space $X$, we put $q^*_Y(f) = \sup\{|f(y)| : y \in Y, q(y) \leq 1\}$ and $q^*(f) = \sup\{|f(x)| : x \in X, q(x) \leq 1\}$, for every linear functional $f$ on $X$. We will make use of the following Theorems.

**Theorem 3.1.** Suppose that $Q$ is a family of seminorms on a real locally convex space $X$ which determines the topology of $X$ and $q \in Q$ is a continuous seminorm and $Y$ is a vector subspace of $X$ such that $Y \cap \{x \in X : q(x) = 0\} = \{0\}$. Let $f$ be a real linear functional on $Y$ such that $q^*_Y(f) < \infty$. Then there exists a continuous linear functional $h$ on $X$ that extends $f$ such that $q^*_Y(f) = q^*(h)$.

**Proof.** If we define $p : X \to \mathbb{R}$ by $p(x) = q^*_Y(f)q(x)$ for each $x \in X$, then we have $p$ is a seminorm on $X$ such that $f(x) \leq p(x)$, for each $x \in Y$. Because, if $x = 0$, clearly $f(x) = 0$ and $0 \leq p(x)$. On the other hand, if $x \in Y$ and $x \neq 0$ then from our assumption, $q(x) \neq 0$ and $q(x/q(x)) = 1$. Therefore, we have $f(x/q(x)) \leq q^*_Y(f)$, then $f(x) \leq q^*_Y(f)q(x) = p(x)$. Since $q$ is continuous, $p$ is also a continuous seminorm, therefore by the Hahn-Banach theorem (Theorem 3.9 in [17]), there exists a linear continuous extension $h$ of $f$ to $X$ that $h(x) \leq p(x)$ for each $x \in X$. Hence, since $X$ is a vector space, we have

$$|h(x)| \leq q^*_Y(f)q(x), (x \in X)$$

and hence, $q^*(h) \leq q^*_Y(f)$. Moreover, since $q^*_Y(f) = \sup\{|f(y)| : q(y) \leq 1\} \leq \sup\{|h(x)| : q(x) \leq 1\} = q^*(h)$, we have $q^*_Y(f) = q^*(h)$. \hfill \Box

**Theorem 3.2.** Suppose that $Q$ is a family of seminorms on a real locally convex space $X$ which determines the topology of $X$ and $q \in Q$ a nonzero continuous seminorm. Let $x_0$ be a point in $X$. Then there exists a continuous linear functional on $X$ such that $q^*(f) = 1$ and $f(x_0) = q(x_0)$.

**Proof.** Let $Y := \{y \in X : q(y) = 0\}$. We consider two cases:

Case 1. Let $x_0 \in Y$.

Since $q$ is continuous, $Y$ is a closed subset of $X$. Indeed, if $x \in \overline{Y}$ and $x_\alpha \in Y$ is a net such that $x_\alpha \to x$. Then we have $q(x) = \lim q(x_\alpha) = 0$, hence $x \in Y$, then $Y$ is a closed. Let $y_0$ be a point in $X \setminus Y$. There exists some $r > 0$ such that $q(y - y_0) > r$ for all $y \in Y$. Suppose that $Z = \{y + \alpha y_0 : \alpha \in \mathbb{R}, y \in Y\}$,
the vector subspace generated by $Y$ and $y_0$. Then we define $h : Z \to \mathbb{R}$ by $h(y + \alpha y_0) = \alpha$. Obviously, $h$ is linear and we have also
\[
|r|h(y + \alpha y_0)| = |r||\alpha| < |\alpha|q(\alpha^{-1}y + y_0) = q(y + \alpha y_0)
\]
for all $y \in Y$ and $\alpha \in \mathbb{R}$. Therefore $h$ is a linear functional on $Z$ that $q_Z^*(h)$ does not exceed $r^{-1}$. Putting $p = r^{-1}q$, we have $p$ is a continuous seminorm such that $h(z) \leq p(z)$ for each $z \in Z$, therefore by the Hahn-Banach theorem (Theorem 3.9 in [17]), there exists a linear continuous extension $L$ of $h$ to $X$ that $L(x) \leq p(x)$ for each $x \in X$. We have also $L(x_0) = h(x_0) = q(x_0) = 0$. Now, since $q_Z^*(h) \neq 0$, we have also $q^*(L) \neq 0$, we can define $f := \frac{L}{q^*(L)}$. Hence, $f$ is a linear continuous functional on $Z$ that $f(x_0) = q(x_0) = 0$ and also $q^*(f) = 1$.

Case 2. Let $x_0 \notin Y$.

Let $Z := \{\alpha x_0 : \alpha \in \mathbb{R}\}$ that is the vector subspace generated by $x_0$. If we define $h(\alpha x_0) = \alpha q(x_0)$ then $h$ is a linear functional on $Z$ that $h(x_0) = q(x_0)$ and also $q_Z^*(h) = 1$. Since $Z \cap Y = \{0\}$, from Theorem 3.1, there exists a continuous linear extension $f$ of $h$ to $X$ such that $q^*(f) = q_Z^*(h) = 1$. Obviously, $f(x_0) = q(x_0)$.

4. Ergodic Retractions for Families of $Q$-Nonexpansive Mappings

Let $Q$ be a family of seminorms on a locally convex space $E$ which determines the topology of $E$. In this section, we study the existence of $Q$-nonexpansive retractions onto the set of common fixed points of a family of $Q$-nonexpansive mappings that commute with the mappings. A $Q$-nonexpansive retraction that commutes with the mappings is usually called an ergodic retraction.

First, we prove the following theorem that extends and generalizes Theorem 2.1 in [22] which is the main result of this section and will be essential in the sequel.

**Theorem 4.1.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Let $C$ be a locally weakly compact and weakly closed convex subset of $E$. Suppose that $S = \{T_i : i \in I\}$ is a family of $Q$-nonexpansive mappings on $C$ such that $\text{Fix}(S) \neq \emptyset$. Consider the following assumption:

(a) $E$ is separated and for every nonempty weakly compact convex $S$-invariant subset $K$ of $C$, $K \cap \text{Fix}(S) \neq \emptyset$,

(b) if $x, y \in C$ and $q(\frac{x+y}{2}) = q(x) = q(y)$ for all $q \in Q$, then $x = y$.

Then, for each $i \in I$, there exists a $Q$-nonexpansive retraction $P_i$ from $C$ onto $\text{Fix}(S)$, such that $P_iT_i = T_iP_i = P_i$ and every weakly closed convex $S$-invariant subset of $C$ is also $P_i$-invariant.
Proof. Let \( C^C \) be the product space with product topology induced by the weak topology on \( C \). Now for a fixed \( \alpha \in I \), consider the following set

\[
\mathcal{R} = \{ T \in C^C : T \text{ is } Q\text{-nonexpansive}, T \circ T_\alpha = T \}
\]

and every weakly closed convex \( S \)-invariant subset of \( C \) is also \( T \)-invariant. From the fact that for each \( z \in \text{Fix}(S) \), the singleton set \( \{ z \} \) is a weakly closed convex \( S \)-invariant subset of \( C \), then for each \( T \in \mathcal{R} \), \( Tz = z \). Fix \( z_0 \in \text{Fix}(S) \) and let, for each \( x \in C \),

\[
C_x := \{ y \in C : q(y - z_0) \leq q(x - z_0), \text{ for all } q \in Q \}.
\]

For all \( x \in C \) and \( T \in \mathcal{R} \), we have that \( T(x) \in C_x \) since \( q(T(x) - z_0) = q(T(x) - T(z_0)) \leq q(x - z_0) \) for all \( q \in Q \). Hence \( \mathcal{R} \subseteq \prod_{x \in C} C_x \), where \( \prod_{x \in C} C_x \) is the Cartesian product of sets \( C_x \) for all \( x \in C \). For each \( q \in Q \), from the fact that for each real number \( \lambda \) the level set \( \{ x \in C : q(x) \leq \lambda \} \) is closed and convex then from Corollary 1.5 (p. 126) in [8] is weakly closed, hence by Proposition 2.5.2 in [2] each seminorm \( q \in Q \) is weakly lower semicontinuous. Because \( C \) is a weakly closed convex subset of \( E \), \( C_x \) is convex and weakly closed for each \( x \in C \). Since \( C \) is locally weakly compact and \( C_x \) is \( \tau_Q \)-bounded, we can conclude that \( C_x \) is weakly compact. By Tychonoff’s theorem, we know that when \( C_x \) is given the weak topology and \( \prod_{x \in C} C_x \) is given the corresponding product topology, \( \prod_{x \in C} C_x \) is compact. Next we prove that \( \mathcal{R} \) is closed in \( \prod_{x \in C} C_x \). Let \( \{ T_\lambda : \lambda \in \Lambda \} \) be a net in \( \mathcal{R} \) which converges to \( T_0 \) in \( \prod_{x \in C} C_x \). Hence if \( z \in \text{Fix}(S) \), \( T_\lambda z = z \), then \( T_0 z = \text{weak} - \lim_\lambda T_\lambda(z) = z \). Since each seminorm \( q \in Q \) is weakly lower semicontinuous, we have 

\[
q(T_0 x - T_0 y) \leq \liminf_\lambda q(T_\lambda x - T_\lambda y) \leq q(x - y),
\]

for each \( x, y \in C \) and \( q \in Q \). Obviously, we have \( T_0 \circ T_\alpha = T_0 \) and every weakly closed convex \( S \)-invariant subset of \( C \) is also \( T_0 \)-invariant. Therefore, \( T_0 \in \mathcal{R} \). Then \( \mathcal{R} \) is closed in \( \prod_{x \in C} C_x \). Since \( \prod_{x \in C} C_x \) is compact, hence \( \mathcal{R} \) is compact. Moreover, \( \mathcal{R} \neq \emptyset \). Indeed, if we define the mapping \( S_n = \frac{1}{n} \sum_0^{n-1} T_\alpha \in \prod_{x \in C} C_x \). Then we have,

\[
\lim_{n \to \infty} S_n T_\alpha - S_n = 0, \quad (2)
\]

on \( C \). Indeed for each \( q \in Q \), we have

\[
q(S_n T_\alpha(z) - S_n(z)) = \frac{1}{n} q(T_\alpha^n(z) - z) = \frac{1}{n} q(T_\alpha^n(z) - T_\alpha^n(z_0) + T_\alpha^n(z_0) - z) \\
\leq \frac{1}{n} q(T_\alpha^n(z) - T_\alpha^n(z_0)) + \frac{1}{n} q(z_0 - z) \\
\leq \frac{2}{n} q(z_0 - z) \to 0,
\]

as \( n \to \infty \), for all \( z \in C \) and from the fact that \( \prod_{x \in C} C_x \) is compact, there exists a (weakly pointwise) convergent subnet \( \{ S_{n(\lambda)} \} \). Hence we can define \( T(x) = \text{weak} - \lim_\lambda S_{n(\lambda)}(x) \). Next, we will show that \( T \in \mathcal{R} \), because each seminorm \( q \in Q \) is weakly lower semicontinuous and \( S_{n(\lambda)} \) is \( Q \)-nonexpansive for each \( \lambda \) then \( T \) is \( Q \)-nonexpansive. From (2), we also have \( T(T_\alpha x) = \text{weak} - \)
\lim_{n(\lambda)} S(T_\alpha x) = \text{weak} - \lim_{\lambda} S_n \lambda x = Tx. \text{ Finally, every weakly closed convex } \mathcal{S}\text{-invariant subset of } C \text{ is } S_n\text{-invariant and hence is } T\text{-invariant. Then } T \in \mathcal{R} \neq \emptyset.

Now let us to define a preorder \( \leq \) in \( \mathcal{R} \) by \( T \leq U \) if \( q(Tx - Ty) \leq q(Ux - Uy) \) for each \( x, y \in C \) and \( q \in Q \), and by using a method similar to Bruck’s method [7], we find a minimal element \( T_{\text{min}} \) in \( \mathcal{R} \). Indeed, Using Zorn’s Lemma, it is enough that we show that each linearly ordered subset of \( \mathcal{R} \) has a lower bound in \( \mathcal{R} \). Hence, let \( \{ A_\lambda \} \) be a linearly ordered subset of \( \mathcal{R} \). Then the family of sets \( \{ T \in \mathcal{R} : T \leq A_\lambda \} \) is a linearly ordered subset of \( \mathcal{R} \) by inclusion. Taking into account the closeness proof of \( \mathcal{R} \) in \( \prod_{x \in C} C_x \), these sets also can accordingly be closed in \( \mathcal{R} \), and hence compact. Then from the finite intersection property, there exists \( R \in \bigcap_\lambda \{ T \in \mathcal{R} : T \leq A_\lambda \} \) with \( R \leq A_\lambda \) for all \( \lambda \). Then each linearly ordered subset of \( \mathcal{R} \) has a lower bound in \( \mathcal{R} \). We have shown until now that there exist a minimal element \( P_\alpha \) in the following sense:

if \( T \in \mathcal{R} \) and \( q(Tx - Ty) \leq q(P_\alpha x - P_\alpha y) \) for each \( x, y \in C \) and \( q \in Q \) then \( q(Tx - Ty) = q(P_\alpha x - P_\alpha y) \). (*)

Next we prove that \( P_\alpha x \in \text{Fix}(\mathcal{S}) \) for every \( x \in C \). For a given \( x \in C \), consider \( K := \{ T(P_\alpha x) : T \in \mathcal{R} \} \). From the fact that \( \mathcal{R} \) is convex and compact, by Proposition 3.3.18 and the Definition 3.3.19 in [18], we conclude that \( K \) is a nonempty weakly compact convex subset of \( C \). Now we have \( \text{STT}_\alpha = ST \) for each \( T \in \mathcal{R} \) hence \( ST \in \mathcal{R} \), i.e., \( K \) is \( \mathcal{S}\)-invariant.

First, consider the case (a): from our assumption \( K \cap \text{Fix}(\mathcal{S}) \neq \emptyset \). Then there exists \( L \in \mathcal{R} \) such that \( L(P_\alpha x) \in \text{Fix}(\mathcal{S}) \). Suppose that \( y = L(P_\alpha x) \). Since \( P_\alpha \cdot L \in \mathcal{R} \) and the set \( \{ y \} \) is \( \mathcal{S}\)-invariant, we have \( P_\alpha(y) = L(y) = y \), and since \( P_\alpha \) is minimal, we have \( q(P_\alpha x - y) = q(P_\alpha x - P_\alpha y) = q(L(P_\alpha x) - L(P_\alpha y)) = q(L(P_\alpha x) - y) = 0 \), for each \( q \in Q \) and since \( E \) is separated, \( P_\alpha x - y = 0 \), hence \( P_\alpha x = y \in \text{Fix}(\mathcal{S}) \) and this holds for each \( x \in C \).

Now, consider the case (b): because \( P_\alpha \in \mathcal{R} \) we have \( T_i P_\alpha T_\alpha = T_i P_\alpha \) for all \( i \). Therefore it is easy to verify that \( T_i P_\alpha \in \mathcal{R} \), for all \( i \). Since \( \mathcal{R} \) is convex, using (*), we have

\[
q(T_i P_\alpha x - z) = q(T_i P_\alpha x - T_i P_\alpha z) = q(P_\alpha x - z) = q\left( \frac{T_i P_\alpha x + P_\alpha x}{2} - z \right),
\]

for each \( x \in C, z \in \text{Fix}(\mathcal{S}) \) and \( i \in I \). Then from our assumption we have \( T_i P_\alpha x = P_\alpha x \) for each \( x \in C \) and \( i \in I \). Hence, \( P_\alpha x \in \text{Fix}(\mathcal{S}) \), for each \( x \in C \).

Since \( P_\alpha \in \mathcal{R} \) and \( \{ P_\alpha x \} \) is \( \mathcal{S}\)-invariant for each \( x \in C \), we conclude that \( P_\alpha^2 = P_\alpha \) and \( P_\alpha T_\alpha = T_\alpha P_\alpha = P_\alpha \).

As a consequence of Theorem 4.1, we prove how we obtain an ergodic retraction by a \( Q\)-nonexpansive retraction.

**Corollary 4.2.** Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \). Let \( C \) be a locally weakly compact and weakly closed convex subset of \( E \). Suppose that \( S = \{ T_i : i \in I \} \) is a
family of $Q$-nonexpansive mappings on $C$ such that $\text{Fix}(S) \neq \emptyset$. If there is a $Q$-nonexpansive retraction $R$ from $C$ onto $\text{Fix}(S)$, then for each $i \in I$, there exists a $Q$-nonexpansive retraction $P_i$ from $C$ onto $\text{Fix}(S)$, such that $P_i T_i = T_i P_i = P_i$, and every weakly closed convex $S \cup \{R\}$-invariant subset of $C$ is also $P_i$-invariant.

**Proof.** Setting $S' := S \cup \{R\}$ and

$$\mathfrak{R}' = \{T \in C^C : T \text{ is } Q\text{-nonexpansive}, T \circ T_\alpha = T\}$$

and every weakly closed convex $S'$-invariant subset of $C$ is also $T$-invariant}, we get that $\text{Fix}(S') = \text{Fix}(S)$ and by replacing $S$ with $S'$ and $\mathfrak{R}$ with $\mathfrak{R}'$ in the proof of Theorem 4.1, we find a minimal element $P_\alpha$ in the sense of (*). Now we have $R \circ T \in \mathfrak{R}'$ for each $T \in \mathfrak{R}'$. Indeed, $R \circ T \circ T_\alpha = R \circ T$ for each $T \in \mathfrak{R}'$ and because $R \in S'$, we have that every weakly closed convex $S'$-invariant subset of $C$ is also $R$-invariant, therefore is $R \circ T$-invariant for each $T \in \mathfrak{R}'$. Hence for each $x \in C$, the set $K = \{T(P_\alpha x) : T \in \mathfrak{R}'\}$ is an $R$-invariant subset of $C$ for each $T \in \mathfrak{R}'$. Therefore from the fact that $R(K) \subset K \cap R(C) = K \cap \text{Fix}(S)$, we have $\text{Fix}(S) \neq \emptyset$. Now by repeating the reasoning used in Theorem 4.1, we will get the desired result. \hfill \square

As an application of Theorem 4.2, we have the following Theorem:

**Corollary 4.3.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Let $C$ be a locally weakly compact and weakly closed convex subset of $E$. Suppose that $S = \{T_i : i \in I\}$ is a family of $Q$-nonexpansive mappings on $C$ such that $\text{Fix}(S) \neq \emptyset$. Consider the following assumption:

(a) $E$ is separated and for every nonempty weakly compact convex $S$-invariant subset $K$ of $C$, $K \cap \text{Fix}(S) \neq \emptyset$,

(b) if $x, y \in C$ and $q(\frac{x+y}{2}) = q(x) = q(y)$ for all $q \in Q$, then $x = y$,

(c) there exists a $Q$-nonexpansive retraction $R$ from $C$ onto $\text{Fix}(S)$.

Let $\{P_i\}_{i \in I}$ be the family of retractions obtained in the above Theorem. Then for each $x \in C$,

$$\{T_i^n x : i \in I, n \in \mathbb{N}\}^{\cap Q} \cap \text{Fix}(S) \subseteq \{P_i(x) : i \in I\}^{\cap Q}.$$

**Proof.** Consider an $\epsilon > 0$ and let $g \in \{T_i^n x : i \in I, n \in \mathbb{N}\}^{\cap Q} \cap \text{Fix}(S)$. Then for each $p \in Q$, there exists $i \in I$ and $n \in \mathbb{N}$ such that $q(T_i^n x - g) < \epsilon$. From our assumptions and using Theorems 4.1 and 4.2, there exists a $Q$-nonexpansive retraction $P_i$ such that $P_i = P_i T_i$, hence we have,

$$q(P_i x - g) = q(P_i T_i^n x - g) \leq q(T_i^n x - g) < \epsilon,$$

then we conclude $g \in \{P_i(x) : i \in I\}^{\cap Q}$. \hfill \square
5. The Weakly Fixed Point Property for Commuting Semigroups

Let $Q$ be a family of seminorms on a locally convex space $E$ which determines the topology of $E$. The goal of this section is to show if $E$ has the weakly fixed point property, then $E$ has the weakly fixed point property for commuting separable semitopological semigroups.

We will make use of the following two Theorems in this section.

**Theorem 5.1.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Let $E$ be Hausdorff and sequentially complete and $C$ be a nonempty closed convex and bounded subset of $E$. If $\{F_n\}$ is a descending sequence of nonempty $Q$-nonexpansive retracts of $C$, then $\bigcap_{n=1}^\infty F_n$ is the fixed point set of some $Q$-nonexpansive mapping $r : C \to C$.

**Proof.** For each $n \in \mathbb{N}$, let $r_n$ be a $Q$-nonexpansive retraction from $C$ onto $F_n$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence such that $\lambda_n > 0$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \lambda_n = 1$ and

$$\lim_{n} \sum_{j=n+1}^\infty \lambda_j = 0.$$  \hspace{1cm} (3)

From our assumptions and as in Lemma 1 in [6], we may assume that $r = \sum_{n=1}^\infty \lambda_n r_n$. Then by lower semicontinuity of each $q \in Q$, we have that $r : C \to C$ is a $Q$-nonexpansive mapping such that $\bigcap_{n=1}^\infty F_n \subseteq \text{Fix}(r)$. Now it suffices to show that $\text{Fix}(r) \subseteq \bigcap_{n=1}^\infty F_n$. Hence consider $x \in \text{Fix}(r)$. Then by lower semicontinuity of each $q \in Q$, we have

$$q(x - r_n(x)) = q(r(x) - r_n(x)) = \left( \sum_{j=1}^\infty \lambda_j [r_j(x) - r_n(x)] \right)$$

$$\leq \sum_{j=1}^\infty \lambda_j q(r_j(x) - r_n(x)),$$

for each $q \in Q$. Because for $1 \leq j < n$, $r_n(x) \in F_n \subseteq F_j$, then we have $r_j r_n(x) = r_n(x)$. Therefore we have

$$q(r_j(x) - r_n(x)) = q(r_j(x) - r_j r_n(x)) \leq q(x - r_n(x))$$;

now for $j = n$, $q(r_j(x) - r_n(x)) = 0$; finally let

$$d_q = q - \text{diam}C = \sup \{q(x - y) : x, y \in C\},$$

then for $j > n$, $q(r_j(x) - r_n(x)) \leq d_q$. Then from (4) we have

$$q(x - r_n(x)) \leq \sum_{j=1}^{n-1} \lambda_j q(x - r_n(x)) + d_q \sum_{j=n+1}^\infty \lambda_j.$$
Because $\sum_{n=1}^{\infty} \lambda_n = 1$, this in turn concludes that

$$q(x - r_n(x)) \leq d_q \frac{\sum_{j=n+1}^{\infty} \lambda_j}{\sum_{j=n}^{\infty} \lambda_j}.$$ 

Then by (3), $r_n(x) \to x$, in $\tau_Q$ when $n \to \infty$. But from the fact that $\{F_n\}$ is descending, therefore $r_n(x) \in F_m$ for each $n \geq m$, and since $F_m$ is the fixed point set of continuous mapping $r_m$, $F_m$ is closed in topology $\tau_Q$. Now since $\lim r_n(x) = x$, we have $x \in F_m$ for $m = 1, 2, 3, \ldots$, therefore $F(r) = \bigcap_n F_n$. □

**Lemma 5.2.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Let $E$ be separated and sequentially complete. If $E$ has the weakly fixed point property, then $E$ has the weakly fixed point property for commuting sequences of $Q$-nonexpansive mappings on $C$.

**Proof.** Since $E$ is separated, $E$ is Hausdorff with respect to the weak topology. Let $C$ be a nonempty weakly compact convex subset of $E$. Since $C$ is weakly compact, then $C$ is weakly closed and by Proposition 2.7 in [17], is weakly bounded hence by Corollary 3.31 in [17] is bounded. Suppose that $\{T_n\}$ be a commuting sequence of $Q$-nonexpansive mappings on $C$. First, we show $\bigcap_{j=1}^{n} \text{Fix}(T_j)$ is a nonempty $Q$-nonexpansive retract of $C$, for all $n \in \mathbb{N}$. The proof is by induction on $n$. For $n = 1$, from Theorem 4.1 and the fact that $E$ has the weakly fixed point property, $\text{Fix}(T_1)$ is a nonempty $Q$-nonexpansive retract of $C$. Now let $\bigcap_{j=1}^{n} \text{Fix}(T_j)$ be a nonempty $Q$-nonexpansive retract of $C$ and $R : C \to \bigcap_{j=1}^{n} \text{Fix}(T_j)$ be a $Q$-nonexpansive retraction. Now we show that $\text{Fix}(T_{n+1}R) = \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$. Obviously we have $\bigcap_{j=1}^{n+1} \text{Fix}(T_j) \subseteq \text{Fix}(T_{n+1}R)$. For the reverse inclusion, let $x \in \text{Fix}(T_{n+1}R)$ then $T_{n+1}R(x) = x$. Since from our assumptions $T_{n+1}$ commutes with $T_1, \ldots, T_n$ and $R(x) \in \bigcap_{j=1}^{n} \text{Fix}(T_j)$, we conclude that $\bigcap_{j=1}^{n} \text{Fix}(T_j)$ is $T_{n+1}$-invariant and hence $x = T_{n+1}(R(x)) \in \bigcap_{j=1}^{n} \text{Fix}(T_j)$). Therefore, $R(x) = x$ and so $x = T_{n+1}(R(x)) = T_{n+1}(x)$. Then $x \in \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$. Thus $\text{Fix}(T_{n+1}R) \subseteq \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$, so $\text{Fix}(T_{n+1}R) = \bigcap_{j=1}^{n+1} \text{Fix}(T_j)$. But, from Theorem 4.1 and the fact that $E$ has the weakly fixed point property, the fixed point set of a $Q$-nonexpansive self mapping of $C$ is a nonempty $Q$-nonexpansive retract of $C$. Therefore, $\bigcap_{j=1}^{n+1} \text{Fix}(T_j)$ is a nonempty $Q$-nonexpansive retract of $C$. Until now we have shown that $\bigcap_{j=1}^{n} \text{Fix}(T_j)$ is a nonempty $Q$-nonexpansive retract of $C$ for each $n \in \mathbb{N}$, thus from Theorem 5.1, we conclude that $\bigcap_{j=1}^{\infty} \text{Fix}(T_j)$ is the fixed point set of some $Q$-nonexpansive mapping $r : C \to C$. So by the weakly fixed point property of $E$, we have proved that $\bigcap_{j=1}^{\infty} \text{Fix}(T_j) = \text{Fix}(r) \neq \emptyset$, this completes the proof. □

Now we prove the main conclusion of this section.
Theorem 5.3. Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \). Let \( E \) be separated and sequentially complete. If \( E \) has the weakly fixed point property, then \( E \) has the weakly fixed point property for commuting separable semitopological semigroups.

Proof. Let \( C \) be a nonempty weakly compact convex subset of \( E \). Let \( S = \{ T_s : s \in S \} \) be a continuous representation of a commuting separable semitopological semigroups \( S \). Suppose that \( \{ s_n \} \) is a countable subset of \( S \) which is dense in \( S \). Hence \( \{ T_{s_n} \} \) is also a commuting sequence. So from Theorem 5.2 and the weakly fixed point property, \( \{ T_{s_n} \} \) has a common fixed point \( z_0 \) in \( C \). But from the fact that \( \{ s_n \} \) is a countable dense subset of \( S \) and \( S \) is a continuous representation of \( S \), we conclude that \( \{ T_s x : s \in S \} \subseteq \{ T_{s_n} x \}^\tau \) for each \( x \in C \). Putting \( x = z_0 \), we have \( \{ T_s z_0 : s \in S \} \subseteq \{ T_{s_n} z_0 \}^\tau = \{ z_0 \} \). Therefore, \( z_0 \) is a common fixed point for \( S \) and so \( \text{Fix}(S) \neq \emptyset \) and the proof is complete. \( \square \)

Open problems: Theorem 5.3 true for amenable semitopological semigroups?

6. Ergodic Retractions for Semigroups in Locally Convex Spaces

Before going to our ergodic theorem, we will need the following two Theorems.

Lemma 6.1. Let \( S \) be a semigroup, \( E \) be a real dual locally convex space with real predual locally convex space \( D \) and \( U \) a convex neighbourhood of 0 in \( D \) and \( p_U \) be the associated Minkowski functional. Let \( f : S \to E \) be a function such that \( \langle x, f(t) \rangle \leq 1 \) for each \( t \in S \) and \( x \in E \). Let \( X \) be a subspace of \( B(S) \) such that the mapping \( t \to \langle x, f(t) \rangle \) be an element of \( X \), for each \( x \in D \). Then, for any \( \mu \in X^* \), there exists a unique element \( F_\mu \in E \) such that \( \langle x, F_\mu \rangle = \mu_1 \langle x, f(t) \rangle \), for all \( x \in D \). Furthermore, if \( 1 \in X \) and \( \mu \) is a mean on \( X \), then \( F_\mu \) is contained in \( \text{co}\{ f(t) : t \in S \}^{w^*} \).

Proof. We define \( F_\mu \) by \( \langle x, F_\mu \rangle = \mu_1 \langle x, f(t) \rangle \) for all \( x \in D \). Obviously, \( F_\mu \) is linear in \( x \). Moreover, from Proposition 3.8 in [17], we have

\[
|\langle x, F_\mu \rangle| = |\mu_1 \langle x, f(t) \rangle| \leq \sup_t |\langle x, f(t) \rangle| \cdot ||\mu|| \leq P_U(x) \cdot ||\mu||, \tag{5}
\]

for all \( x \in D \). Let \( (x_\alpha) \) be a net in \( D \) that converges to \( x_0 \). Then by (5) we have

\[
|\langle x_\alpha, F_\mu \rangle - \langle x_0, F_\mu \rangle| = |\langle x_\alpha - x_0, F_\mu \rangle| \leq P_U(x_\alpha - x_0) \cdot ||\mu||,
\]

taking limit, since from Theorem 3.7 in [17], \( P_U \) is continuous, we have \( F_\mu \) is continues on \( D \), hence \( F_\mu \in E \).

Now, let \( 1 \in X \) and \( \mu \) be a mean on \( X \). Then, there exists a net \( \{ \mu_\alpha \} \) of finite means on \( X \) such that \( \{ \mu_\alpha \} \) converges to \( \mu \) with the weak* topology on \( X^* \). We may consider that
\begin{equation*}
\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} \delta_{t_{\alpha,i}}.
\end{equation*}

Therefore,

\begin{equation*}
\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle = \left\langle x, \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \right\rangle, \quad (\forall x \in D, \forall \alpha),
\end{equation*}

then we have

\begin{equation*}
F_{\mu_\alpha} = \sum_{i=1}^{n_\alpha} \lambda_{\alpha,i} f(t_{\alpha,i}) \in \co\{f(t) : t \in S\}, (\forall \alpha),
\end{equation*}

now since,

\begin{equation*}
\langle x, F_{\mu_\alpha} \rangle = (\mu_\alpha)_t \langle x, f(t) \rangle \rightarrow \mu_t \langle x, f(t) \rangle = \langle x, f(t) \rangle, \quad (x \in D),
\end{equation*}

\{F_{\mu_\alpha}\} converges to \( F_\mu \) in the weak* topology. Hence

\begin{equation*}
F_\mu \in \overline{\co\{f(t) : t \in S\}^w*},
\end{equation*}

we can write \( F_\mu \) by \( \int f(t) d\mu(t). \)

\textbf{Lemma 6.2.} Let \( S \) be a semigroup, \( C \) a closed convex subset of a real locally convex space \( E \). Let \( B \) be a base at 0 for the topology consisting of convex, balanced sets. Let \( Q = \{q_V : V \in B\} \) which \( q_V \) is the associated Minkowski functional with \( V \). Let \( S = \{T_s : s \in S\} \) be a \( Q \)-nonexpansive representation of \( S \) as mappings from \( C \) into itself and \( X \) be a subspace of \( B(S) \) such that \( 1 \in X \) and \( \mu \) be a mean on \( X \). If we write \( T_\mu x \) instead of \( \int T_t x \; d\mu(t) \), then the following hold.

\begin{enumerate}
\item If the mapping \( t \rightarrow \langle T_t x - T_t y, x^* \rangle \) is an element of \( X \) for each \( t \in S \), \( x, y \in C \) and \( x^* \in E^* \) then \( T_\mu \) is a \( Q \)-nonexpansive mapping from \( C \) into \( C \),
\item if the mapping \( t \rightarrow \langle T_t x, x^* \rangle \) is an element of \( X \) for each \( x \in C \) and \( x^* \in E^* \) then \( T_\mu x = x \) for each \( x \in \text{Fix}(S) \),
\item If moreover \( E \) is a real dual locally convex space with real predual locally convex space \( D \) and \( C \) a \( w^* \)-closed convex subset of \( E \) and \( U \) a convex neighbourhood of 0 in \( D \) and \( p_U \) is the associated Minkowski functional. Let the mapping \( t \rightarrow \langle z, T_t x \rangle \) is an element of \( X \) for each \( x \in C \) and \( z \in D \) then \( T_\mu x \in \overline{\co\{T_t x : t \in S\}^w} \) for each \( x \in C \),
\item if the mapping \( t \rightarrow \langle T_t x, x^* \rangle \) is an element of \( X \) for each \( x \in C \) and \( x^* \in E^* \) and \( X \) is \( r_s \)-invariant for each \( s \in S \) and \( \mu \) is right invariant, then \( T_{\mu t} T_t = T_{\mu t} \) for each \( t \in S \),
\item if \( a \in E \) is an \( Q \)-attractive point of \( S \) and the mapping \( t \rightarrow \langle a - T_t x, x^* \rangle \) is an element of \( X \) for each \( t \in S \), \( x \in C \) and \( x^* \in E^* \) then \( a \) is an \( Q \)-attractive point of \( T_\mu \).
\end{enumerate}
Proof. (i) Let \( x, y \in C \) and \( V \in \mathcal{B} \). By Proposition 3.33 in [17], the topology on \( E \) induced by \( Q \) is the original topology on \( E \). By Theorem 3.7 in [17], \( q_V \) is a continuous seminorm and from Theorem 1.36 in [19], \( q_V \) is a nonzero seminorm because if \( x \not\in V \) then \( q_V(x) \geq 1 \), hence from Theorem 3.2, there exists a functional \( x^*_V \in X^* \) such that \( q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x^*_V \rangle \) and \( q_V^*(x^*_V) = 1 \), and since from Theorem 3.7 in [17], \( q_V(z) \leq 1 \) for each \( z \in V \), we conclude that \( \langle z, x^*_V \rangle \leq 1 \) for all \( z \in V \). Therefore from Theorem 3.8 in [17], \( \langle z, x^*_V \rangle \leq q_V(z) \) for all \( z \in E \). Hence from the fact that the function \( t \to \langle T_t x - T_t y, x^* \rangle \) is an element of \( X \) for each \( t \in S \) and \( x^* \in E^* \), we have

\[
q_V(T_\mu x - T_\mu y) = \langle T_\mu x - T_\mu y, x^*_V \rangle = \mu(t(x, x^*) - (x, x^*))
\]

then we have

\[
q_V(T_\mu x - T_\mu y) \leq q_V(x - y),
\]

for all \( V \in \mathcal{B} \).

(ii) Let \( x \in Fix(S) \) and \( x^* \in E^* \). Then we have

\[
\langle T_\mu x, x^* \rangle = \mu(t(x, x^*) = \langle x, x^* \rangle
\]

(iii) this assertion concludes from Theorem 6.1.

(iv) for this assertion, note that

\[
\langle T_\mu(T_s x), x^* \rangle = \mu(t(x, x^*) = \langle x, x^* \rangle = \langle T_\mu x, x^* \rangle,
\]

(v) Let \( x \in C \) and \( V \in \mathcal{B} \). From Theorem 3.2, there exists a functional \( x^*_V \in X^* \) such that \( q_V(a - T_\mu x) = \langle a - T_\mu x, x^*_V \rangle \) and \( q_V^*(x^*_V) = 1 \). Since from Theorem 3.7 in [17], \( q_V(z) \leq 1 \) for each \( z \in V \), we conclude that \( \langle z, x^*_V \rangle \leq 1 \) for all \( z \in V \). Therefore from Theorem 3.8 in [17], \( \langle z, x^*_V \rangle \leq q_V(z) \) for all \( z \in E \). Hence from the fact that the function \( t \to \langle a - T_t x, x^* \rangle \) is an element of \( X \) for each \( t \in S \) and \( x^* \in E^* \), we have

\[
q_V(a - T_\mu x) = \langle a - T_\mu x, x^*_V \rangle = \mu(t(x, x^*) - (x, x^*))
\]

then we have

\[
q_V(a - T_\mu x) \leq q_V(a - x),
\]

for all \( V \in \mathcal{B} \). \( \square \)

Now we exhibit our ergodic theorem.
Theorem 6.3. Let $S$ be a semigroup, $C$ a locally weakly compact and closed convex subset of a real locally convex space $E$. Let $B$ be a base at 0 for the topology consisting of convex, balanced sets. Let $Q = \{q_V : V \in B\}$ which $q_V$ is the associated Minkowski functional with $V$. Let $S = \{T_s : s \in S\}$ be a right amenable $Q$-nonexpansive semigroup on $C$ such that $\text{Fix}(S) \neq \emptyset$. Consider the following assumption:

(a) $E$ is separated and for every nonempty weakly compact convex $S$-invariant subset $K$ of $C$, $K \cap \text{Fix}(S) \neq \emptyset$,

(b) if $x, y \in C$ and $q_V(\frac{x+y}{2}) = q_V(x) = q_V(y)$ for all $V \in B$, then $x = y$.

Then there exists a $Q$-nonexpansive retraction $P$ from $C$ onto $\text{Fix}(S)$, such that $P T_t = T_t P = P$ for every $t \in S$, and every weakly closed convex $S$-invariant subset of $C$ is also $P$-invariant.

Proof. Consider the following set:

$$\mathcal{R} = \{T \in CC^C : T \text{ is } Q\text{-nonexpansive, } T \circ T_t = T, \forall t \in S\}$$

and every weakly closed convex $S$-invariant subset of $C$ is also $T$-invariant.

Replacing this set by $\mathcal{R}$ in the proof of Theorem 4.1, we can repeat the argument used in the proof of Theorem 4.1 to get the desired result. Of course, from Theorem 6.2, we have $T_\mu \in \mathcal{R} \neq \emptyset$ and $C_x$ be as in the proof of Theorem 4.1, then $\mathcal{R} \subseteq \prod_{x \in C} C_x$. As in the proof of Theorem 4.1, $\prod_{x \in C} C_x$ is compact when $C_x$ is given the weak topology and $\prod_{x \in C} C_x$ is given the corresponding product topology, and $\mathcal{R}$ is compact in $\prod_{x \in C} C_x$. Using the preorder $\preceq$ in $\mathcal{R}$ defined by $T \preceq U$ if $q(Tx - Ty) \leq q(Ux - Uy)$ for each $x, y \in C$ and $q \in Q$, there exist a minimal element $P$ in the following sense:

if $T \in \mathcal{R}$ and $q(Tx - Ty) \leq q(Px - Py)$ for each $x, y \in C$ and $q \in Q$ then $q(Tx - Ty) = q(Px - Py)$. As in the proof of Theorem 4.1, we have $Px \in \text{Fix}(S)$ for every $x \in C$, $P^2 = P$ and $P T_t = T_t P = P$. □

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