Quantum sliding mode control via error sliding surface

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Abstract
In this paper, a new quantum sliding mode control, for improving the performance of the two-level quantum sliding mode control systems with bounded uncertainties, is introduced. The presented quantum sliding surface is based on the error which occurs between the predetermined sliding mode and the system state. The control objective is to derive the system state to reach the sliding mode domain and then maintain its motion on it. For this purpose, we use the sliding mode control method and periodic projective measurements. A theorem for facilitating the presented method is proved. The simulated example shows that both the reaching time to the sliding mode and the control amplitude are significantly decreased, which demonstrate the effectiveness and validity of the presented method.

Keywords
Quantum sliding mode, sliding condition, quantum sliding mode domain, projective Measurement

1. Introduction
Over the last three decades, the quantum control theory has been developed rapidly. Since the control of quantum phenomena has been one of the main goals of many researches in physics and chemistry from the beginning, the quantum control theory has had valuable developments in physical chemistry (Dantus and Lozovoy, 2004; Rabitz, 2003; Rabitz et al., 2000; Rice and Zhao, 2000; Shapiro and Brumer, 2003), atomic and molecular physics (Bonacic-Koutecky and Mitric, 2005; Chu, 2002), quantum optics (Van Handel et al., 2005; Wiseman and Milburn, 1993), and quantum computer design (Mahler et al., 2002). In quantum control theory the main issue is the controllability of the system. In the real world, every quantum system confronts uncertainties, disturbances, and incomplete knowledge which will affect the control results. From a control point of view, all of these uncertainties can be considered as uncertainties in the control field, Hamiltonian system, and quantum states. To overcome these uncertainties, the robust quantum control approach is considered. In recent years, some of the methods in classical robust control have been extended into the quantum domain. James et al. (2008) have formulated and solved a quantum linear stochastic problem using the $H^\infty$ method. To analyze the robustness of the quantum feedback network, D’Helon and James (2006) have used the small gain theorem. Sampled-data control combined with robust control has been used in some other papers (Dong and Petersen, 2011; Dong et al., 2012a, 2012b, 2013). In order to robustly control a two-level quantum system with bounded uncertainties in the Hamiltonian system a sampled-data control approach based on the sliding mode design has been used. In this method, the control law has been designed off-line and then used on-line (Dong and Petersen, 2011). Dong et al., in two different papers have used the sampled data control method for robust decoherence control of a single qubit which has operator errors. To improve the control performance the sliding mode domain has been used (Dong et al., 2012a, 2012b). A two-level quantum system with two classes of uncertainties involving the Hamiltonian system and the coupling strength of the system–environment interaction has been considered. As the system has uncertainties, the sliding mode
domain has been used to construct a robust control (Dong et al., 2013). In Dong and Petersen (2009), two approaches with different kinds of sliding mode, have been introduced to enhance the system robustness against some specific kinds of uncertainties in the Hamiltonian system. Dong and Petersen (2012a, 2012b), have presented a robust quantum control method which was a combination of sliding mode control, Lyapunov control, and periodic projective measurement to overcome some different kinds of uncertainties again. In this paper, we consider a two-level system which has uncertainties in the Hamiltonian system, and we present a new sliding surface and combine it with periodic projective measurement to improve the system robustness performance. In fact, this method presents a significant improvement of the Dong and Petersen method (Dong and Petersen, 2012a, 2012b).

This paper is organized in four sections: in Section 2, the problem formulation and sliding mode domain is presented. Section 3 describes the control process; and to show the effectiveness of the presented method an example is solved in Section 4.

2. Problem formulation and sliding mode

In this section, the two-level quantum sliding mode control, based on the new sliding surface and the sliding domain, is presented.

2.1. Two-level quantum system

A pure state two-level quantum system can be described by the following Schrödinger equation

\[
\frac{\text{d}}{\text{d}t} |\psi(t)\rangle = \left( H_0 + H_\Delta + \sum_{k=x,y,z} u_k(t) H_k \right) |\psi(t)\rangle,
\]

\[
|\psi(0)\rangle = | \varphi_0 \rangle,
\]

where \( \hbar \) is the reduced Planck constant and set to be 1 in this paper, and \( H_\Delta \) is the Hamiltonian uncertainty and can be denoted as \( H_\Delta = \varepsilon_x(t) H_x + \varepsilon_y(t) H_y + \varepsilon_z(t) H_z \). Furthermore, we assume that the uncertainties are bounded. Therefore, let \( \sqrt{\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2} \leq \varepsilon \) for \( \varepsilon \geq 0 \). Finally, in the above Schrodinger equation the Hamiltonian matrices \( H_x, H_y, \) and \( H_z \) are multiples of Pauli matrices which are defined as follows

\[
H_x = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_y = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad H_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Also, in this paper, we set \( H_0 = H_z \).

Every physical state \( |\psi\rangle \) of a two-level quantum system can be written as a superposition of two states \( |0\rangle \) and \( |1\rangle \) as

\[
|\psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1,
\]

where \( |0\rangle \) and \( |1\rangle \) are the eigenstates of the free Hamiltonian. In spherical coordinate the Equation (2) will be represented as

\[
|\psi\rangle = \cos \left( \frac{\theta}{2} \right) |0\rangle + \sin \left( \frac{\theta}{2} \right) e^{i\phi} |1\rangle, \quad \phi \in [0, 2\pi], \theta \in [0, \pi].
\]

Also, in a physical point of view any two states with different global phases represent the same state. In other words, any two states \( |\varphi_1\rangle \) and \( |\varphi_2\rangle = e^{i\phi} |\varphi_1\rangle \), for all \( \phi \in \mathbb{R} \), represent the same state (D’Alessandro, 2008).

2.2. Sliding surface and sliding mode domain

In the variable structure method, sliding mode control plays an important role (Utkin, 1977). Assume that \( |\varphi_j\rangle \) is the eigenstate that corresponds to the eigenvalue \( \lambda_j \).

In quantum sliding mode control we can consider one of the eigenstates of the free Hamiltonian \( H_0 \) as the sliding mode. Let \( |\varphi_j\rangle \) be considered as the sliding mode. The error which occurs between \( |\varphi_j\rangle \) and the system state \( |\psi\rangle \) can be defined as \( |\epsilon\rangle = |\psi\rangle - |\varphi_j\rangle \). Now, we consider the new sliding surface as follows

\[
s(|\psi\rangle, t) = \frac{1}{2} |||\epsilon\rangle||^2 = \frac{1}{2} \left( |\langle \psi | \varphi_j \rangle + \langle \varphi_j | \psi \rangle | \right)
\]

\[
= 1 - \frac{1}{2} \left( |\langle \psi | \varphi_j \rangle + \langle \varphi_j | \psi \rangle | \right) = 0.
\]

When the system state is on the sliding surface, assume that the system has desired dynamics. Assume that the initial state of the system is on the sliding surface. So, we have

\[
s(|\varphi_0\rangle, t_0) = 1 - \frac{1}{2} \left( |\langle \varphi_0 | \varphi_j \rangle + \langle \varphi_j | \varphi_0 \rangle | \right) = 0.
\]

Now, we can easily show that the system will remain on the surface under action of only the free Hamiltonian. If there is no Hamiltonian except the free one, the system state is given by \( |\psi\rangle = e^{-iH_0t} |\varphi_0\rangle \). Thus

\[
s(|\psi\rangle, t) = 1 - \frac{1}{2} \left( |\langle \psi | \varphi_j \rangle + \langle \varphi_j | \psi \rangle | \right)
\]

\[
= 1 - \frac{1}{2} \left( |\langle \varphi_0 | e^{-iH_0t} | \varphi_j \rangle + \langle \varphi_j | e^{-iH_0t} | \varphi_0 \rangle | \right)
\]

\[
= 1 - \frac{1}{2} \left( |\langle \varphi_0 | \varphi_j \rangle + \langle \varphi_j | \varphi_0 \rangle | \right) = 0.
\]