On the spectrum of a class of distance-transitive graphs

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Abstract

Let $\Gamma = \text{Cay}(\mathbb{Z}_n, S_k)$ be the Cayley graph on the cyclic additive group $\mathbb{Z}_n$ ($n \geq 4$), where $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are the inverse-closed subsets of $\mathbb{Z}_n - \{0\}$ for any $k \in \mathbb{N}$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. In this paper, we will show that $\chi(\Gamma) = \omega(\Gamma) = k + 1$ if and only if $k + 1 | n$. Also, we will show that if $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\text{Aut}(\Gamma) \cong \mathbb{Z}_2 \wr_1 \text{Sym}(k + 1)$ where $I = \{1, \ldots, k + 1\}$ and in this case, we show that $\Gamma$ is an integral graph.

Keywords: Cayley graph, distance-transitive, wreath product
Mathematics Subject Classification : 05C15, 05C50
DOI:10.5614/ejgta.2017.5.1.7

1. Introduction

In this paper, a graph $\Gamma = (V, E)$ always means a simple connected graph with $n$ vertices (without loops, multiple edges and isolated vertices), where $V = V(\Gamma)$ is the vertex set and $E = E(\Gamma)$ is the edge set. The size of the largest clique in the graph $\Gamma$ is denoted by $\chi(\Gamma)$ and the size of the largest independent sets of vertices by $\alpha(\Gamma)$. A graph $\Gamma$ is called a vertex-transitive graph if for any $x, y \in V$ there is some $\pi$ in $\text{Aut}(\Gamma)$, the automorphism group of $\Gamma$, such that $\pi(x) = y$. Let $\Gamma$ be a graph, the complement $\bar{\Gamma}$ of $\Gamma$ is the graph whose vertex set is $V(\Gamma)$ and whose edges are the pairs of nonadjacent vertices of $\Gamma$. It is well known that for any graph $\Gamma$, $\text{Aut}(\Gamma) = \text{Aut}(\bar{\Gamma})$
If $\Gamma$ is a connected graph and $\partial(u,v)$ denotes the distance in $\Gamma$ between the vertices $u$ and $v$, then for any automorphism $\pi$ in $\text{Aut}(\Gamma)$ we have $\partial(u,v) = \partial(\pi(u),\pi(v))$.

Let $k$ be a positive integer, a $k$-colouring of a graph $\Gamma$ is a mapping $f : V(\Gamma) \rightarrow \{1, \ldots, k\}$ such that $f(x) \neq f(y)$ for any two adjacent vertices $x$ and $y$ in $\Gamma$, and if such a mapping exists we say that $\Gamma$ is $k$-colorable. The chromatic number $\chi(\Gamma)$ of $\Gamma$ is the minimum number $k$ such that $\Gamma$ is $k$-colorable. Let $\Gamma$ be a graph and $I(\Gamma)$ denote the set of all independent sets of the graph $\Gamma$. A fractional colouring of a graph $\Gamma$ is a weight function $\mu : I(\Gamma) \rightarrow [0,1]$ such that for any vertex $x$ of $\Gamma$, $\sum_{I \in I(\Gamma)} \mu(I) \geq 1$, and if such a weight function exists we say that $\Gamma$ is fractional colouring. The fractional chromatic number of a graph $\Gamma$ is denoted by $\chi_f(\Gamma)$ and defined in [9, Page 134]. Also a fractional clique of a graph $\Gamma$ is denoted by $\psi_f(\Gamma)$ and defined in [9, Page 134].

Let $\Upsilon = \{\gamma_1, \ldots, \gamma_{k+1}\}$ be a set and $K$ be a group then we write $\text{Fun}(\Upsilon, K)$ to denote the set of all functions from $\Upsilon$ into $K$, we can turn $\text{Fun}(\Upsilon, K)$ into a group by defining a product:

$$(fg)\gamma = f\gamma g\gamma \quad \text{for all} \quad f, g \in \text{Fun}(\Upsilon, K) \quad \text{and} \quad \gamma \in \Upsilon,$$

where the product on the right is in $K$. Since $\Upsilon$ is finite, the group $\text{Fun}(\Upsilon, K)$ is isomorphic to $K^{k+1}$ (a direct product of $k + 1$ copies of $K$) via the isomorphism $f \rightarrow (f(\gamma_1), \ldots, f(\gamma_{k+1}))$. Let $H$ and $K$ be groups and suppose $H$ acts on the nonempty set $\Upsilon$. Then the wreath product of $K$ by $H$ with respect to this action is defined to be the semidirect product $\text{Fun}(\Upsilon, K) \rtimes H$ where $H$ acts on the group $\text{Fun}(\Upsilon, K)$ via

$$f^x\gamma = f^{\gamma^{-1}} \quad \text{for all} \quad f \in \text{Fun}(\Upsilon, K), \gamma \in \Upsilon \quad \text{and} \quad x \in H.$$  

We denote this group by $K^{wr}_{\Upsilon}H$. Consider the wreath product $G = K^{wr}_{\Upsilon}H$. If $K$ acts on a set $\Delta$ then we can define an action of $G$ on $\Delta \times \Upsilon$ by

$$(\delta, \gamma)^{(f, h)} = (\delta^{f(\gamma)}, h) \quad \text{for all} \quad (\delta, \gamma) \in \Delta \times \Upsilon,$$

where $(f, h) \in \text{Fun}(\Upsilon, K) \rtimes H = K^{wr}_{\Upsilon}H$ [6].

Eigenvalues of an undirected graph $\Gamma$ are the eigenvalues of an arbitrary adjacency matrix of $\Gamma$. Harary and Schwenk [10] defined $\Gamma$ to be integral, if all of its eigenvalues are integers. For a survey of integral graphs see [3]. In [2] the number of integral graphs on $n$ vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes, see [1].

Let $G$ be a finite group and $S$ a subset of $G$ that is closed under taking inverses and does not contain the identity. A Cayley graph $\Gamma = Cay(G, S)$ is a graph whose vertex-set and edge-set are defined as follows:

$$V(\Gamma) = G; \quad E(\Gamma) = \{\{x, y\} \mid x^{-1}y \in S\}.$$  

It is well known that every Cayley graph is vertex-transitive.

For any graph $\Gamma$, $\omega(\Gamma) = \chi(\Gamma) \geq 2$. Also it is well known that for bipartite graphs $\omega(\Gamma) = \chi(\Gamma) = 2$. Let $\Gamma$ be the $\text{Cay}(\mathbb{Z}_n, S_k)$ where $\mathbb{Z}_n$ ($n \geq 4$), is the cyclic additive group with identity $\{0\}$, and for any $k \in \mathbb{N}$, $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$, $S_1 = \{1, n-1\}, \ldots, S_k = S_{k-1} \cup \{k, n-k\}$ are inverse-closed subsets of $\mathbb{Z}_n - \{0\}$. In this paper we will show that $\chi(\Gamma) = \omega(\Gamma) = k + 1$ if and only if $k + 1 | n$, also we show that if $n$ is an even integer and $k = \frac{n}{2} - 1$ then $\text{Aut}(\Gamma) \cong \mathbb{Z}_{2^{w_{\Upsilon}1}}\text{Sym}(k+1)$, where $I = \{1, \ldots, k + 1\}$. 

64