AN IMPLICIT ALGORITHM FOR FINDING A FIXED POINT OF A Q-NONEXPANSIVE MAPPING IN LOCALLY CONVEX SPACES

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ABSTRACT. Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. We introduce an implicit method for finding an element of the set of fixed points of a $Q$-nonexpansive mapping. Then we prove the convergence of the proposed implicit scheme to a fixed point of the $Q$-nonexpansive mapping in $\tau_Q$. For this purpose, some new concepts are devised in locally convex spaces.

1. Introduction

Let $C$ be a nonempty closed and convex subset of a Banach space $E$ and $E^*$ be the dual space of $E$. Let $\langle.,.\rangle$ denote the pairing between $E$ and $E^*$. The normalized duality mapping $J : E \rightarrow E^*$ is defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}$$

for all $x \in E$. In this investigation we study duality mappings for locally convex spaces that will be denoted by $J_q$ for a seminorm $q$.

Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$ that will be denoted by $\tau_Q$. Let $C$ be a nonempty closed and convex subset of $E$. A mapping $T$ of $C$ into itself is called $Q$-nonexpansive if $q(Tx - Ty) \leq q(x - y)$, for all $x, y \in C$ and $q \in Q$, and a mapping $f$ is a $Q$-contraction on $E$ if $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in E$ such that $0 \leq \beta < 1$.

In this paper we introduce the following general implicit algorithm for finding an element of the set of fixed points of a $Q$-nonexpansive mapping. On the other hand, our goal is to prove that there exists a sunny $Q$-nonexpansive retraction $P$ of $C$ onto $\text{Fix}(T)$ and $x \in C$ such that the following sequence $\{z_n\}$ converges to $Px$ in $\tau_Q$.

$$z_n = \epsilon_n fz_n + (1 - \epsilon_n)Tz_n \quad (n \in I),$$

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where \( f \) is a \( Q \)-contraction and \( T \) is a \( Q \)-nonexpansive mapping. To receive to the aim, some new concepts in locally convex spaces will be devised. For example, some new results of Hahn Banach theorem and Banach contraction principle will be generalized to locally convex spaces.

2. PRELIMINARIES

Suppose that \( Q \) is a family of seminorms on a locally convex space \( E \) which determines the topology of \( E \) that will be denoted by \( \tau_Q \). Let \( D \) be a subset of \( B \) where \( B \) is a subset of a locally convex space \( E \) and let \( P \) be a retraction of \( B \) onto \( D \), that is, \( Px = x \) for each \( x \in D \). Then \( P \) is said to be sunny, if for each \( x \in B \) and \( t \geq 0 \) with \( Px + t(x - Px) \in B \), \( P(Px + t(x - Px)) = Px \). A subset \( D \) of \( B \) is said to be a sunny \( Q \)-nonexpansive retract of \( B \) if there exists a sunny \( Q \)-nonexpansive retraction \( P \) of \( B \) onto \( D \). We know that for a Banach space if \( E \) is smooth and \( P \) is a retraction of \( B \) onto \( D \), then \( P \) is sunny and nonexpansive if and only if for each \( x \in B \) and \( z \in D \),

\[
\langle x - Px, J(z - Px) \rangle \leq 0.
\]

(2.1)

For more details, see [6]. In this paper, we prove inequality (2.1) for locally convex spaces.

Suppose that \( Q \) is a family of seminorms on a locally convex space \( X \) which determines the topology of \( X \). Recall the following definitions:

1. The locally convex topology \( \tau_Q \) is separated if and only if the family of seminorms \( Q \) possesses the following property: for each \( x \in X \setminus \{0\} \) there exists \( q \in Q \) such that \( q(x) \neq 0 \) or equivalently

\[
\bigcap_{q \in Q} \{x \in X : q(x) = 0\} = \{0\} \quad (\text{see } [3]),
\]

2. let \( E \) be a locally convex topological vector space over \( \mathbb{R} \) or \( \mathbb{C} \). If \( U \subset E \), then the polar of \( U \), denoted by \( U^* \), is the set

\[
\{f \in E^* : |f(x)| \leq 1, \forall x \in U\}.
\]

3. SOME RESULTS OF HAHN BANACH THEOREM

Suppose that \( Q \) is a family of seminorms on a locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \) is a seminorm. Let \( Y \) be a subset of \( X \), we put \( q^*_Y(f) = \sup\{|f(y)| : y \in Y, q(y) \leq 1\} \) and \( q^*(f) = \sup\{|f(x)| : x \in X, q(x) \leq 1\} \), for every linear functional \( f \) on \( X \). Observe that, for each \( x \in X \) that \( q(x) \neq 0 \) and \( f \in X^* \), then \( |\langle x, f \rangle| \leq q(x)q^*(f) \). We will make use of the following Theorems.

**Theorem 3.1.** Suppose that \( Q \) is a family of seminorms on a real locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \) is a continuous seminorm and \( Y \) is a vector subspace of \( X \) such that \( Y \cap \{x \in X : q(x) = 0\} = \{0\} \). Then \( X \) is solid.
0\} = \{0\}. Let \( f \) be a real linear functional on \( Y \) such that \( q_Y^*(f) < \infty \). Then there exists a continuous linear functional \( h \) on \( X \) that extends \( f \) such that \( q_Y^*(f) = q^*(h) \).

**Proof.** If we define \( p : X \to \mathbb{R} \) by \( p(x) = q_Y^*(f)q(x) \) for each \( x \in X \), then we have \( p \) is a seminorm on \( X \) such that \( f(x) \leq p(x) \), for each \( x \in Y \). Because, if \( x = 0 \), clearly \( f(x) = 0 \) and \( 0 \leq p(x) \). On the other hand, if \( x \in Y \) and \( x \neq 0 \) then from our assumption, \( q(x) \neq 0 \) and \( q(\frac{x}{q(x)}) = 1 \). Therefore, we have \( f(\frac{x}{q(x)}) \leq q_Y^*(f) \), then \( f(x) \leq q_Y^*(f)q(x) = p(x) \). Since \( q \) is continuous, \( p \) is also a continuous seminorm, therefore by the Hahn-Banach theorem (Theorem 3.9 in \([5]\)) there exists a linear continuous extension \( h \) of \( f \) to \( X \) such that \( h(x) \leq p(x) \) for each \( x \in X \). Hence, since \( X \) is a vector space, we have

\[
|h(x)| \leq q_Y^*(f)q(x), (x \in X)
\]

and hence, \( q^*(h) \leq q_Y^*(f) \). Moreover, since \( q_Y^*(f) = \sup\{|f(x)| : x \in Y, q(x) \leq 1\} \leq \sup\{|h(x)| : q(x) \leq 1\} = q^*(h) \), we have \( q_Y^*(f) = q^*(h) \).

**Theorem 3.2.** Suppose that \( Q \) is a family of seminorms on a real locally convex space \( X \) which determines the topology of \( X \) and \( q \in Q \) a nonzero continuous seminorm. Let \( x_0 \) be a point in \( X \). Then there exists a continuous linear functional on \( X \) such that \( q^*(f) = 1 \) and \( f(x_0) = q(x_0) \).

**Proof.** Let \( Y := \{y \in X : q(y) = 0\} \). We consider two cases:

Case 1. Let \( x_0 \in Y \). Since \( q \) is continuous, \( Y \) is a closed subset of \( X \). Indeed, if \( x \in Y \) and \( x_0 \in Y \) is a net such that \( x_0 \to x \). Then we have \( q(x) = \lim q(x_0) = 0 \), hence \( x \in Y \). Then \( Y \) is a closed. Let \( y_0 \) be a point in \( X \setminus Y \). There exists some \( r > 0 \) such that \( q(y - y_0) > r \) for all \( y \in Y \). Suppose that \( Z = \{y + \alpha y_0 : \alpha \in \mathbb{R}, y \in Y\} \), the vector subspace generated by \( Y \) and \( y_0 \). Then we define \( h : Z \to \mathbb{R} \) by \( h(y + \alpha y_0) = \alpha \). Obviously, \( h \) is linear and we have also \( r|h(y + \alpha y_0)| = r|\alpha| < |\alpha|q(\alpha^{-1}y + y_0) = q(y + \alpha y_0) \) for all \( y \in Y \) and \( \alpha \in \mathbb{R} \). Therefore \( h \) is a linear functional on \( Z \) that \( q_Z^*(h) \) does not exceed \( r^{-1} \). Putting \( p = r^{-1}q \), we have \( p \) is a continuous seminorm such that \( h(z) \leq p(z) \) for each \( z \in Z \), therefore by the Hahn-Banach theorem (Theorem 3.9 in \([5]\)), there exists a linear continuous extension \( L \) of \( h \) to \( X \) such that \( L(x) \leq p(x) \) for each \( x \in X \). We have also \( L(x_0) = h(x_0) = q(x_0) = 0 \). Now, since \( q_Z^*(h) \neq 0 \), we have also \( q^*(L) \neq 0 \), we can define \( f := \frac{L}{q^*(L)} \). Hence, \( f \) is a linear continuous functional on \( Z \) that \( f(x_0) = q(x_0) = 0 \) and also \( q^*(f) = 1 \).

Case 2. Let \( x_0 \notin Y \). Let \( Z := \{\alpha x_0 : \alpha \in \mathbb{R}\} \) that is the vector subspace generated by \( x_0 \). If we define \( h(\alpha x_0) = \alpha q(x_0) \) then \( h \) is a linear functional on \( Z \) that \( h(x_0) = q(x_0) \) and also \( q_Z^*(h) = 1 \). Since \( Z \cap Y = \{0\} \), from Theorem 3.1, there exists a continuous linear extension \( f \) of \( h \) to \( X \) such that \( q^*(f) = q_Z^*(h) = 1 \). Obviously, \( f(x_0) = q(x_0) \).
Now we are ready to define our $q$-duality mapping:

Suppose that $Q$ is a family of seminorms on a real locally convex space $X$ which determines the topology of $X$, $q \in Q$ is a continuous seminorm and $X^*$ is the dual space of $X$. A multivalued mapping $J_q : X \to 2^{X^*}$ defined by

$$J_q x = \{ j \in X^* : \langle x, j \rangle = q(x)^2 = q^*(j)^2 \},$$

is called $q$-duality mapping. Obviously, $J_q(-x) = -J_q(x)$. $J_q x \neq \emptyset$. Indeed, let $x \in X$, if $q(x) = 0$, $j = 0$ is in $J_q x$, and in other words, if $q(x) \neq 0$, from Theorem 3.2, there exists a linear functional $f \in X^*$ such that $q^*(f) = 1$ and $\langle x, f \rangle = q(x)$. Putting $j := q(x)f$, we have

$$\langle x, j \rangle = \langle x, q(x)f \rangle = q(x)\langle x, f \rangle = q(x)^2,$$

and we have also,

$$q^*(j) = \sup\{|j(y)| : y \in X, q(y) \leq 1\} = \sup\{|q(x)f(y)| : y \in X, q(y) \leq 1\}
= q(x) \sup\{|f(y)| : y \in X, q(y) \leq 1\} = q(x)q^*(f) = q(x).$$

4. Main result

**Lemma** 4.1. Suppose that $Q$ is a family of seminorms on a Hausdorff and complete locally convex space $E$ which determines the topology of $E$. Let $\phi_q : E \to (-\infty, \infty]$ be a bounded below lower semicontinuous function for each $q \in Q$. Suppose that $\{x_n\}$ is a sequence in $E$ such that $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $q \in Q$. Then $\{x_n\}$ converges to a point $v \in E$ and for each $q \in Q$,

$$q(x_n - v) \leq \phi_q(x_n) - \phi_q(v)$$

for all $n \in \mathbb{N}_0$.

**Proof.** Since $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$, for each $n \in \mathbb{N}_0$ and $q \in Q$, then we have $\{\phi_q(x_n)\}$ is a decreasing sequence for each $q \in Q$. Moreover, for $m \in \mathbb{N}_0$,

$$\sum_{n=0}^{m} q(x_n - x_{n+1}) = q(x_0 - x_1) + q(x_1 - x_2) + \ldots + q(x_m - x_{m+1})$$
\[ \leq \phi_q(x_0) - \phi_q(x_{m+1}) \]
\[ \leq \phi_q(x_0) - \inf_{n \in \mathbb{N}_0} \phi_q(x_n). \]

Letting $m \to \infty$, we have

$$\sum_{n=0}^{\infty} q(x_n - x_{n+1}) < \infty.$$
for each \( q \in Q \), then \( \lim_{n} q(x_n - x_{n+1}) = 0 \) for each \( q \in Q \). This implies that \( \{x_n\} \) is a left Cauchy sequence in \( E \). Because \( E \) is Hausdorff and complete, there exists a unique \( v \in E \) such that \( \lim_{n \to \infty} x_n = v \). Let \( m, n \in \mathbb{N}_0 \) with \( m > n \). Then for each \( q \in Q \)

\[
q(x_n - x_m) \leq \sum_{i=n}^{m-1} q(x_i - x_{i+1}) \\
\leq \phi_q(x_n) - \phi_q(x_m).
\]

Letting \( m \to \infty \), since \( \phi_q \) is lower semicontinuous for each \( q \in Q \) and from Theorem 1.4 in [3] each \( q \in Q \) is continuous, then for each \( q \in Q \), we conclude that

\[
q(x_n - v) \leq \phi_q(x_n) - \lim_{m \to \infty} \phi_q(x_m) \leq \phi_q(x_n) - \phi_q(v)
\]

for all \( n \in \mathbb{N}_0 \). \( \square \)

Now we state an extension of Banach Contraction Principle to locally convex spaces and we called it Banach \( Q \)-Contraction Principle.

**Theorem 4.2.** (Banach \( Q \)-Contraction Principle) Suppose that \( Q \) is a family of seminorms on a separated and complete locally convex space \( E \) which determines the topology of \( E \) and \( T : E \to E \) a \( Q \)-contraction mapping with Lipschitz constant \( k \in (0,1) \). Then we have the following:

(a) There exists a unique fixed point \( v \in E \).

(b) For arbitrary \( x_0 \in E \), the Picard iteration process defined by

\[
x_{n+1} = T(x_n), n \in \mathbb{N}_0,
\]

converges to \( v \).

(c) \( q(x_n - v) \leq (1 - k^{-1}k^n q(x_0 - x_1)) \) for all \( n \in \mathbb{N}_0 \) and \( q \in Q \).

**Proof.** (a) For each \( q \in Q \), let \( \phi_q : E \to \mathbb{R}^+ \) be a functions defined by

\[
\phi_q(x) = (1 - k)^{-1}q(x - Tx),
\]

for each \( x \in E \). From Theorem 1.4 in [3], since each \( q \) is continuous, \( \phi_q \) is also a continuous function. From the fact that \( T \) is a \( Q \)-contraction mapping, for each \( q \in Q \) we have

\[
q(Tx - T^2x) \leq kq(x - Tx), x \in E, \tag{4.1}
\]

which conclude that

\[
q(x - Tx) - kq(x - Tx) \leq q(x - Tx) - q(Tx - T^2x).
\]
Hence
\[ q(x - T x) \leq \frac{1}{1 - k}[q(x - T x) - q(T x - T^2 x)] \]
\[ \leq \frac{1}{1 - k}[q(x - T x) - q(T x - T^2 x)], \]
for each \( q \in Q \), therefore
\[ q(x - T x) \leq \phi_q(x) - \phi_q(T x). \] (4.2)
Consider an arbitrary element \( x \) in \( X \) and define the sequence \( x_n \) in \( E \) by
\[ x_n = T^n x, n \in \mathbb{N}. \]
From (4.2), we have
\[ q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1}), n \in \mathbb{N}, \]
for each \( q \in Q \) and since \( E \) is Hausdorff, it follows from Lemma 4.1 that there exists an element \( v \in E \) such that
\[ \lim_{n \to \infty} x_n = v, \]
and
\[ q(x_n - v) \leq \phi_q(x_n), n \in \mathbb{N}_0, \] (4.3)
for each \( q \in Q \). Since for example, from page 3 in [3], \( T \) is continuous and \( x_{n+1} = T x_n \), it follows that \( v = T v \). Suppose that \( z \) is another fixed point for \( T \). If for each \( q \in Q \) we have \( q(v - z) = 0 \), then from the fact that \( E \) is separated, we have \( v = z \). On the other hands, let \( 0 < q(v - z) \) for some \( q \in Q \), then we have
\[ 0 < q(v - z) = q(T v - T z) \leq k q(v - z) < q(v - z), \]
that is a contradiction. Hence \( T \) has unique fixed point \( v \in E \).

(b) This assertion follows from part (a).

(c) From (4.1) we have that \( \phi_q(x_n) \leq k^n \phi_q(x_0) \) for each \( q \in Q \). This implies from (4.3) that \( q(x_n - v) \leq k^n \phi_q(x_0) \) for each \( q \in Q \). \( \square \)

**Lemma 4.3.** Let \( E \) be a locally convex space. Then for \( x, y \in E \) with \( q(x) \neq 0 \), the following are equivalent:

(a) \( q(x) \leq q(x + ty) \) for all \( t > 0 \) that \( q(x + ty) \neq 0 \) and \( q \in Q \).

(b) There exists \( j_q \in J_q \) such that \( \langle y, j_q \rangle \geq 0 \) for each \( q \in Q \).

Proof. (a) \( \Rightarrow \) (b). For \( t > 0 \), let \( f_t \in J_q(x + ty) \) and define \( g_t = \frac{f_t}{q^*(f_t)} \). Hence \( q^*(g_t) = 1 \). Since \( g_t \in q^*(f_t)^{-1} J_q(x + ty) \) and \( q^*(g_t) = 1 \), we have
\[ q(x) \leq q(x + ty) = q^*(f_t)^{-1} \langle x + ty, f_t \rangle \]
\[ = \langle x + ty, g_t \rangle = \langle x, g_t \rangle + t \langle y, g_t \rangle \]
\[ \leq q(x) + t \langle y, g_t \rangle. \] (4.4)
By Theorem 3.26 in [5], the Banach-Alaoglu theorem (which states that the polar $U^\circ$ is weak*ly-compact for every neighborhood of zero in $E$). Putting $U = \{ x \in E : q(x) \leq 1 \}$, we have $\{ g_t \} \subset U^\circ$ hence, the net $\{ g_t \}$ has a limit point $g \in E^*$ such that $q^*(g) \leq 1$ and from (4.4) we have that $\langle x, g \rangle \geq q(x)$ and $\langle y, g \rangle \geq 0$. Observe that

$$q(x) \leq \langle x, g \rangle \leq q(x)q^*(g) \leq q(x),$$

which gives that $\langle x, g \rangle = q(x)$ and $q^*(g) = 1$. Set $j_q = gq(x)$, then $j_q \in J_q x$ and $\langle y, j_q \rangle \geq 0$.

(b) $\Rightarrow$ (a). Suppose for $x, y \in X$ with $q(x) \neq 0$ and $q \in Q$, there exists $j_q \in J_q x$ such that $\langle y, j_q \rangle \geq 0$. Hence for $t > 0$ that $q(x + ty) \neq 0$,

$$q(x)^2 = \langle x, j_q \rangle \leq \langle x + ty, j_q \rangle = \langle x + ty, j_q \rangle \leq q(x + ty)q(x),$$

which implies that $q(x) \leq q(x + ty)$.

**Theorem 4.4.** Let $C$ be a nonempty convex subset of a separated locally convex space $X$ and $D$ a nonempty subset of $C$. Let $J_q : E \rightarrow E^*$ be single valued for every $q \in Q$. Let $(C - C) \cap \{ x, q(x) = 0 \} = \{ 0 \}$. If $P$ is a retraction of $C$ onto $D$ such that for each $q \in Q$,

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad (x \in C, y \in D), \quad (4.5)$$

then $P$ is sunny $Q$-nonexpansive. Conversely, if $P$ is sunny $Q$-nonexpansive and $(D - D) \cap \{ x, q(x) = 0 \} = \{ 0 \}$, then (4.5) holds.

**Proof.** First we show $P$ is sunny: For $x \in C$, put $x_t := Px + t(x - Px)$ for each $t > 0$. Since $C$ is convex, we conclude that $x_t \in C$ for each $t \in (0, 1]$. Hence, from (4.5), we have

$$\langle x - Px, J_q(Px - Px_t) \rangle \geq 0 \text{ and } \langle x_t - Px_t, J_q(Px_t - Px) \rangle \geq 0. \quad (4.6)$$

Because $x_t - Px = t(x - Px)$ and $\langle t(x - Px), J_q(Px - Px_t) \rangle \geq 0$, we have

$$\langle x_t - Px, J_q(Px - Px_t) \rangle \geq 0. \quad (4.7)$$

Combining (4.6) and (4.7), we have

$$q(Px - Px_t)^2 = \langle Px - x_t + x_t - Px_t, J_q(Px - Px_t) \rangle$$

$$= -\langle x_t - Px, J_q(Px - Px_t) \rangle + \langle x_t - Px_t, J_q(Px - Px_t) \rangle \leq 0.$$
Conversely, suppose that $P$ is the sunny $Q$-nonexpansive retraction and $x \in C$. Then $Px \in D$ and there exists a point $z \in D$ such that $Px = z$. Putting $M := \{z + t(x - z) : t \geq 0\}$ we conclude $M$ is a nonempty convex set. Since $P$ is sunny, i.e., $Pv = z$, for each $v \in M$ we have

$$q(y - z) = q(Py - Pv) \leq q(y - v) = q(y - z + t(z - x))$$

for all $y \in D$. Hence, from Lemma 4.3, we have

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad (x \in C, y \in D).$$

□

To prove Theorem 4.9, we need to prove the following tree corollaries of Theorem 6.5.3 in [6].

**Corollary 4.5.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Let $K$ be a convex subset of $E$. Let $x \in E$ and $x_0 \in K$ such that $(E - \{x\}) \cap \{y \in E : q(y) = 0\} = \{0\}$, for each $q \in Q$. Then the following are equivalent, for each $q \in Q$:

1. $q(x_0 - x) = \inf\{q(x - y) : y \in K\}$;
2. there exists an $L \in E^*$ such that $q^*(L) = 1$ and

$$\inf\{L(y - x) : y \in K\} = q(x_0 - x);$$

3. there exists an $L \in E^*$ such that $q^*(L) = 1$ and

$$\inf\{L(y) : y \in K\} = L(x_0),$$

and

$$L(x_0 - x) = q(x_0 - x).$$

**Proof.** By Theorem 6.5.3 in [6], the corollary holds. We just need to show $q^*(L) = 1$ and $L$ is continuous in the assertion $(1) \Rightarrow (2)$. We know that, for each $\epsilon > 0$ there exists a point $y_0 \in K$ such that

$$q(y_0 - x) \leq q(x_0 - x) + \epsilon \leq L(y_0 - x),$$

then by our assumption, we have

$$1 \leq L(\frac{y_0 - x}{q(y_0 - x)}) \leq q^*(L) \leq 1,$$

hence $q^*(L) = 1$. To show the continuity of $L$, let $x_\alpha$ be a net in $E$ such that $x_\alpha \to z_0$, then

$$\langle x_\alpha - z_0, L \rangle \leq q(x_\alpha - z_0),$$

then $L$ is continuous. □
**Corollary 4.6.** Suppose that $Q$ is a family of seminorms on a real locally convex space $E$ which determines the topology of $E$. Let $K$ be a convex subset of $E$. Let $x \in E$ and $x_0 \in K$ such that $(E - \{x\}) \cap \{y \in E : q(y) = 0\} = \{0\}$, for each $q \in Q$. Then the following are equivalent:

1. $q(x_0 - x) = \inf\{q(x - y) : y \in K\}$, for each $q \in Q$;
2. there exists an $f \in J_q(x - x_0)$ such that

$$\langle x_0 - y, f \rangle \geq 0,$$

for every $y \in K$ for each $q \in Q$.

**Proof.** (1)⇒(2): Let $q(x_0 - x) = \inf\{q(x - y) : y \in K\}$, for each $q \in Q$. Then from theorem 4.5, there exists an $L \in E^*$ such that $q^*(L) = 1$ and

$$\inf\{L(y - x) : y \in K\} = q(x_0 - x).$$

Set $f = -q(x_0 - x)L$. Therefore, since $q(x_0 - x) \leq L(x_0 - x) \leq q(x_0 - x)$, we have

$$f(x - x_0) = -q(x_0 - x)L(x - x_0) = q^2(x_0 - x) = q^*(f).$$

Then $f \in J_q(x - x_0)$. Hence, we have, for every $y \in K$,

$$f(x_0 - y) = f(x_0 - x + x - y) = -q^2(x_0 - x) - q(x_0 - x)L(y - x)$$

$$= -q^2(x_0 - x) + q(x_0 - x)L(y - x)$$

$$\geq -q^2(x_0 - x) + q^2(x_0 - x) = 0.$$

(2)⇒(1): If $q(x_0 - x) = 0$ then (1) holds. Hence, assume that $q(x_0 - x) \neq 0$. Therefore, we have

$$q^2(x_0 - x) = \langle x - x_0, f \rangle = \langle x - y + y - x_0, f \rangle$$

$$\leq q(x - y)q^*(f) + \langle y - x_0, f \rangle \leq q(x - y)q^*(f) = q(x - y)q(x_0 - x),$$

hence we have

$$q(x_0 - x) \leq q(x - y),$$

for all $y \in K$. Then, we have $q(x_0 - x) = \inf\{q(x - y) : y \in K\}$, for each $q \in Q$. \hfill \square

**Corollary 4.7.** Suppose that $Q$ is a family of seminorms on a locally convex space $E$ which determines the topology of $E$. Suppose that $C$ is a nonempty closed convex subset of $E$. Let $C_0 \subseteq C$ and $P$ be a sunny $Q$-nonexpansive retraction of $C$ onto $C_0$. Let $J_q$ be single valued duality mapping for each $q \in Q$. Then for any $x \in C$ and $y \in C_0$,

$$\langle x - Px, J_q(y - Px) \rangle \leq 0.$$
Proof. Suppose $x \in C$ and $y \in C_0$. Set $x_t = Px + t(x - Px)$ for each $0 \leq t \leq 1$. Then we have $x_t \in C$ and $q(y - Px) = q(Py - Px_t) \leq q(y - x_t)$ for each $q \in Q$. By Corollary 4.6 we have
\[
\langle Px - x_t, J_q(y - Px) \rangle \leq 0,
\]
hence,
\[
\langle x - Px, J_q(y - Px) \rangle \leq 0.
\]

Theorem 4.8. Let $E$ be a locally convex space and $J_q : E \to E^{\ast}$ a single-valued duality mapping. Then $J_q$ is continuous from $\tau_Q$ to weak* topology.

Proof. We show that if $x_\alpha \to x$ in $\tau_Q$ then $J_q x_\alpha \to J_q x$ in the weak* topology.

First, assume that $q(x) = 0$, then $J_q x = 0$. We need to show that $J_q x_\alpha \to 0$ in the weak* topology. But, we have $q(x_\alpha) = q^\ast(J_q x_\alpha)$, and since from Theorem 1.4 in [3] each $q \in Q$ is continuous, we have $0 = \lim_{\alpha} q(x_\alpha) = \lim_{\alpha} q^\ast(J_q x_\alpha)$, hence, $\lim_{\alpha} q^\ast(J_q x_\alpha) = 0$, therefore, $\lim_{\alpha} \langle y, J_q x_\alpha \rangle = 0$ for every $y \in E$ that $q(y) \leq 1$. Then for every $y \in E$, $\lim_{\alpha} \langle y, J_q x_\alpha \rangle = 0$. Indeed, if $q(y) > 1$, then $q(\frac{y}{q(y)}) \leq 1$, therefore $\lim_{\alpha} \langle y, f_\alpha \rangle = 0$, then for every $y \in E$, $\lim_{\alpha} \langle y, J_q x_\alpha \rangle = 0$. Hence $J_q x_\alpha \to 0$ in the weak* topology.

Second, assume that $q(x) \neq 0$. Set $f_\alpha^q := J_q x_\alpha$. Then
\[
\langle x_\alpha, f_\alpha^q \rangle = q(x_\alpha)q^\ast(f_\alpha^q), \quad q(x_\alpha) = q^\ast(f_\alpha^q).
\]
Because $(x_\alpha)$ is convergent in $\tau_Q$, $(x_\alpha)$ is bounded with respect to $\tau_Q$, then we can conclude that $f_\alpha^q$ is bounded in $E^\ast$. Indeed, since $q^\ast(f_\alpha^q) = q(x_\alpha)$ and we know from Definition 1.1 in [4] that $E^\ast$ is a l.c.s that the seminorms \(p_y : y \in E\) which $p_y(x^*) = |\langle y, x^* \rangle|$, define the weak* topology on it, hence,
\[
p_y(f_\alpha^q) = |\langle y, f_\alpha^q \rangle| \leq q(y)q^\ast(f_\alpha^q) = q(y)q(x_\alpha)
\]
when $q(y) \neq 0$, and in other words, when $q(y) = 0$, from the definition of $J_q x_\alpha$ and $q^\ast(f_\alpha^q)$ we have that
\[
|\langle y, f_\alpha^q \rangle| \leq q^\ast(f_\alpha^q) = q(x_\alpha)
\]
hence, from (4.8) and (4.9) and from the boundedness concept in locally convex spaces, page 3 in [3], and since $(x_\alpha)$ is bounded then $(f_\alpha^q)$ is bounded in $E^\ast$ for each $q \in Q$. From (4.8) and (4.9), we can select an upper bound $M_{q,y} \neq 0$ related to each $q \in Q$ and $y \in E$ for $(f_\alpha^q)$ in the weak* topology. Putting
\[
U_q = \{ z \in E : q(z) < 1 \}, \quad (\frac{1}{M_{q,y}} f_\alpha^q) \subset U_q^\circ \text{ for each } q \in Q \text{ and } y \in E
\]
where $U_q^\circ$ is the polar of $U_q$. Then from Theorem 3.26 (Banach-Alaoglu) in [5], there exists a subnet \((\frac{1}{M_{q,y}} f_\alpha^q)_{\beta}\) of \((\frac{1}{M_{q,y}} f_\alpha^q)\) such that $\frac{1}{M_{q,y}} f_\alpha^q \to f \in U_q^\circ$ in the weak* topology.
We know that the function $q^*$ on $E^*$ is lower semicontinuous in weak* topology. Indeed, if $g_\beta \in \{f \in E^* : q^*(f) \leq \alpha\}$ such that $g_\beta \to g$ in the weak* topology, then equivalently, from Definition 1.1 in [4] and page 3 in [3], $p_y(g_\beta - g) \to 0$ for each $y \in E$, hence $\langle y, g_\beta - g \rangle \to 0$, therefore, $\langle y, g_\beta \rangle \to \langle y, g \rangle$, hence $q^*(g) \leq \alpha$ i.e. $g \in \{f \in E^* : q^*(f) \leq \alpha\}$ then from Proposition 2.5.2 in [1], $q^*$ is lower semicontinuous in the weak* topology on $E^*$. Therefore, we have

$$q^*(f) \leq \liminf_{\beta} q^*(\frac{1}{M_{q,y}}f_{\alpha,\beta}^q) = \frac{1}{M_{q,y}} \liminf_{\beta} q(x_{\alpha,\beta}) = \frac{1}{M_{q,y}} q(x).$$ (4.10)

Because $\langle x, M_{q,y} f - f_{\alpha,\beta}^q \rangle \to 0$ and $\langle x - x_{\alpha,\beta}, f_{\alpha,\beta}^q \rangle \to 0$, indeed from pages 125 and 126 [4], the weak topology on a l.c.s is coarser than the original topology, hence, from the fact that $x_\alpha \to x$ in the original topology, therefore $x_\alpha \to x$ in the weak topology. Now, we have

$$|\langle x, M_{q,y} f \rangle - q(x_{\alpha,\beta})^2| = |\langle x, M_{q,y} f \rangle - \langle x_{\alpha,\beta}, f_{\alpha,\beta}^q \rangle|$$

$$\leq |\langle x, M_{q,y} f \rangle - f_{\alpha,\beta}^q| + |\langle x - x_{\alpha,\beta}, f_{\alpha,\beta}^q \rangle| \to 0,$$

and since from Theorem 1.4 in [3] each $q \in Q$ is continuous, we have

$$\langle x, M_{q,y} f \rangle = q(x)^2.$$

Since $q(x) \neq 0$, we have

$$q(x)^2 = \langle x, M_{q,y} f \rangle \leq q^*(M_{q,y} f)q(x).$$

Thus, using (4.10), we have $\langle x, M_{q,y} f \rangle = q(x)^2, q(x) = q^*(M_{q,y} f)$. Therefore, $M_{q,y} f = J_q x$.

In the next theorem, we prove an existence theorem of a sunny $Q$-nonexpansive retract.

**Theorem 4.9.** Suppose that $Q$ is a family of seminorms on a real separated and complete locally convex space $E$ which determines the topology of $E$. Let $(C - C) \cap \{x, q(x) = 0\} = \{0\}$, for each $q \in Q$. Suppose that $C$ is a nonempty closed convex and bounded subset of $E$ such that every sequence in $C$ has a convergent subsequence. Let $T$ be a $Q$-nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $J_q : E \to E^*$ be single valued for every $q \in Q$. Then $\text{Fix}(T)$ is a sunny $Q$-nonexpansive retract of $C$ and the sunny $Q$-nonexpansive retraction of $C$ onto $\text{Fix}(T)$ is unique.

**Proof.** By theorem 4.2 as in the proof of step 1 in Theorem 4.11 that we will prove, we put a sequence $\{z_n\}$ in $C$ as follows:

$$z_n = \frac{1}{n}x + (1 - \frac{1}{n})Tz_n \quad (n \in \mathbb{N}),$$ (4.11)
where, $x \in C$ is fixed. Then we have
\[
\lim_{n \to \infty} q(z_n - T z_n) = 0.
\] (4.12)
for each $q \in Q$. Because, from the fact that $C$ is bounded and $T$ is $Q$-nonexpansive, we have
\[
\lim_{n} q(Tz_n - z_n) = \lim_{n} q(Tz_n - \frac{1}{n}x - (1 - \frac{1}{n})Tz_n)
= \lim_{n} \frac{1}{n}q(x - Tz_n) = 0,
\]
for each $q \in Q$.

Next, we show that the sequence $\{z_n\}$ converges to an element of Fix(T). In the other words, we show that the limit set of $\{z_n\}$ (denoted by $\mathcal{S}\{z_n\}$) is a subset of Fix(T). For each $z \in \text{Fix}(T)$, $n \in \mathbb{N}$ and $q \in Q$, we have
\[
\langle z_n - x, J_q(z_n - z) \rangle \leq 0.
\]
Indeed, we have for each $z \in \text{Fix}(T)$,
\[
\langle z_n - x, J_q(z_n - z) \rangle = \langle \frac{1}{n}x + (1 - \frac{1}{n})Tz_n - x, J_q(z_n - z) \rangle
= (n - 1)\langle Tz_n - z_n, J_q(z_n - z) \rangle
= (n - 1)\langle Tz_n - Tz, J_q(z_n - z) \rangle
+ (n - 1)\langle z - z_n, J_q(z_n - z) \rangle
\leq (n - 1)(q(Tz_n - Tz)q(z_n - z) - q(z_n - z)^2)
\leq (n - 1)(q(z_n - z)^2 - q(z_n - z)^2) = 0.
\]
Furthermore, we have, for each $q \in Q$, $\lim_{n} q(Tz_n - z_n) = 0$. Because from the fact that $C$ is bounded and $T$ is $Q$-nonexpansive, we have that $\{x - Tz_n\}$ is bounded then
\[
\lim_{n} q(Tz_n - z_n) = \lim_{n} q(Tz_n - \frac{1}{n}x - (1 - \frac{1}{n})Tz_n)
= \lim_{n} \frac{1}{n}q(x - Tz_n) = 0
\] (4.13)
for each $q \in Q$. From our assumption, $\{z_n\}$ has a subsequence converges to a point in $C$. Let $\{z_{n_i}\}$ and $\{z_{n_j}\}$ be subsequences of $\{z_n\}$ such that $\{z_{n_i}\}$ and $\{z_{n_j}\}$ converge to $y$ and $z$, respectively. Therefore, $y, z \in \text{Fix}(T)$. Because from (4.13), for each $q \in Q$, we have
\[
q(y - Ty) \leq q(y - z_{n_i}) + q(z_{n_i} - T z_{n_i}) + q(T z_{n_i} - Ty)
\leq 2q(y - z_{n_i}) + q(z_{n_i} - T z_{n_i}).
\]
Taking limit, since $E$ is separated, we have $y \in \text{Fix}(T)$ and similarly $z \in \text{Fix}(T)$. Further, we have

$$ \langle y - x, J_q(y - z) \rangle = \lim_{i \to \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle \leq 0. \quad (4.14) $$

Indeed, from the fact that $J_q$ is single valued and since from Theorem 4.8, $J_q$ is continuous from $\tau_Q$ to weak* topology, we have

$$ |\langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle| $$

$$ = |\langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(z_{n_i} - z) \rangle + \langle y - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle| $$

$$ \leq |\langle z_{n_i} - y, J_q(z_{n_i} - z) \rangle| + |\langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle| $$

$$ \leq q(z_{n_i} - y)q(z_{n_i} - z) + |\langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle| $$

$$ \leq q(z_{n_i} - y)M_q + |\langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle|, $$

where $M_q$ is an upper bound for $\{z_{n_i} - z\}_{i \in \mathbb{N}}$ for each $q \in Q$. Hence, for each $q \in Q$, we have

$$ \langle y - x, J_q(y - z) \rangle = \lim_{i \to \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle \leq 0. $$

Similarly $\langle z - x, J_q(z - y) \rangle \leq 0$ and therefore $y = z$. Indeed, since $J_q(y - z) = -J_q(y - z)$ we have

$$ q(y - z)^2 = \langle y - z, J_q(y - z) \rangle = \langle y - x, J_q(y - z) \rangle + \langle x - z, J_q(y - z) \rangle $$

$$ = \langle y - x, J_q(y - z) \rangle + \langle z - x, J_q(z - y) \rangle \leq 0, $$

then $q(y - z) = 0$, for each $q \in Q$, and since $E$ is separated, $y = z$. Thus, $\{z_n\}$ converges to an element of $\text{Fix}(T)$.

Therefore, a mapping $P$ of $C$ into itself can be defined by $P x = \lim_{n} z_n$. Then we have, for each $z \in \text{Fix}(T)$,

$$ \langle x - P x, J_q(z - P x) \rangle = \lim_{n \to \infty} \langle z_n - x, J_q(z_n - z) \rangle \leq 0. \quad (4.15) $$

It follows from Theorem 4.4 that $P$ is a sunny $Q$-nonexpansive retraction of $C$ onto $\text{Fix}(T)$. Suppose $R$ is another sunny $Q$-nonexpansive retraction of $C$ onto $\text{Fix}(T)$. Then, from Corollary 4.7, we have, for each $x \in C$ and $z \in \text{Fix}(T)$,

$$ \langle x - Rx, J_q(z - Rx) \rangle \leq 0. \quad (4.16) $$

Putting $z = Rx$ in (4.15) and $z = P x$ in (4.16), we have $\langle x - P x, J_q(Rx - P x) \rangle \leq 0$ and $\langle x - Rx, J_q(P x - Rx) \rangle \leq 0$ and hence $\langle Rx - P x, J_q(Rx - P x) \rangle \leq 0$. Then we have $q^2(Rx - P x) \leq 0$ for each $q \in Q$ and since $E$ is separated, this implies $Rx = P x$. This completes the proof. \hfill \Box
Proposition 4.10. Suppose that $Q$ is a family of seminorms on a separated locally convex space $E$ which determines the topology of $E$. Then
\[ q(x)^2 - q(y)^2 \geq 2\langle x - y, j \rangle \]
for all $x, y \in E$ and $j \in J_qy$ such that $q(y) \neq 0$.

Proof. Let $j \in J_qx$, $x \in E$. Then
\[
q(y)^2 - q(x)^2 - 2\langle y - x, j \rangle = q(x)^2 + q(y)^2 - 2\langle y, j \rangle = q(x)^2 + q(y)^2 - 2q(x)q(y) \geq (q(x) - q(y))^2 \geq 0.
\]

Theorem 4.11. Suppose that $Q$ is a family of seminorms on a real separated and complete locally convex space $E$ which determines the topology of $E$ and $C$ be a nonempty closed convex and bounded subset of $E$ such that every sequence in $C$ has a convergent subsequence. Suppose that $T$ is a $Q$-nonexpansive mapping from $C$ into itself such that Fix($T$) $\neq \emptyset$. Assume that $J_q$ is single valued for each $q \in Q$. Let $(C - C) \cap \{x, q(x) = 0\} = \{0\}$ for each $q \in Q$. Suppose that $f$ is an $Q$-contraction on $C$. Let $\epsilon_n$ be a sequence in $(0, 1)$ such that $\lim \epsilon_n = 0$. Then there exists a unique $x \in C$ and sunny $Q$-nonexpansive retraction $P$ of $C$ onto Fix($T$) such that the following net \( \{z_n\} \) generated by
\[
z_n = \epsilon_n f z_n + (1 - \epsilon_n) T z_n \quad (n \in I),
\]
converges to $Px$.

Proof. Since $f$ is a $Q$-contraction, there exists $0 \leq \beta < 1$ such that $q(f(x) - f(y)) \leq \beta q(x - y)$ for each $x, y \in E$ and $q \in Q$. We divide the proof into five steps.

Step 1. The existence of $z_n$ which satisfies (4.17).

Proof. This follows immediately from the fact that for every $n \in I$, the mapping $N_n$ given by
\[
N_n x := \epsilon_n f x + (1 - \epsilon_n) T x \quad (x \in C),
\]
is a $Q$-contraction. To see this, put
\[
\beta_n = (1 + \epsilon_n(\beta - 1)), \quad 0 \leq \beta_n < 1 \quad (n \in \mathbb{N}).
\]
Then we have,
\[
q(N_n x - N_n y) \leq \epsilon_n q(f x - f y) + (1 - \epsilon_n) q(T x - T y)
\]
\[
\leq \epsilon_n \beta q(x - y) + (1 - \epsilon_n) q(x - y)
\]
\[
= (1 + \epsilon_n(\beta - 1)) q(x - y) = \beta_n q(x - y).
\]
Therefore, by Theorem 4.2, there exists a unique point \( z_n \in C \) such that \( N_n z_n = z_n \).

Step 2. \( \lim_{n} q(Tz_n - z_n) = 0 \) for each \( q \in Q \).

Proof. Since \( C \) is bounded, we have that \( \{fz_n - Tz_n\} \) is bounded then
\[
\lim_{n} q(Tz_n - z_n) = \lim_{n} q(Tz_n - \epsilon_n fz_n - (1 - \epsilon_n)Tz_n) = \lim_{n} \epsilon_n q(fz_n - Tz_n) = 0
\]
for each \( q \in Q \).

Step 3. \( \mathcal{G}\{z_n\} \subseteq \text{Fix}(T) \), where \( \mathcal{G}\{z_n\} \) denotes the set of \( \tau_Q \)-limit points of subsequences of \( \{z_n\} \).

Proof. Let \( z \in \mathcal{G}\{z_n\} \), and let \( \{z_{n_k}\} \) be a subsequence of \( \{z_n\} \) such that \( z_{n_k} \to z \). For each \( q \in Q \), we have
\[
q(Tz - z) \leq q(Tz - Tz_{n_k}) + q(Tz_{n_k} - z_{n_k}) + q(z_{n_k} - z) \\
\leq 2q(z_{n_k} - z) + q(Tz_{n_k} - z_{n_k}),
\]
then by Step 2,
\[
q(Tz - z) \leq 2 \lim_{k} q(z_{n_k} - z) + \lim_{j} q(Tz_{n_k} - z_{n_k}) = 0,
\]
for each \( q \in Q \), hence \( q(Tz - z) = 0 \) for each \( q \in Q \) and since \( E \) is separated we have \( z \in \text{Fix}(T) \).

Step 4. There exists a unique sunny \( Q \)-nonexpansive retraction \( P \) of \( C \) onto \( \text{Fix}(T) \) and \( x \in C \) such that
\[
K := \lim_{n} \sup \langle x - Px, J_q(z_n - Px) \rangle \leq 0. \quad (4.18)
\]

Proof. We know from Theorem 4.8 that \( J_q \) is continuous from \( \tau_Q \) to weak* topology, then by Theorem 4.9 there exists a unique sunny \( Q \)-nonexpansive retraction \( P \) of \( C \) onto \( \text{Fix}(T) \). Theorem 4.2 guarantees that \( fP \) has a unique fixed point \( x \in C \). We show that
\[
K := \lim_{n} \sup \langle x - Px, J_q(z_n - Px) \rangle \leq 0.
\]

Note that, from the definition of \( K \) and our assumption that every sequence in \( C \) has a convergent subsequence, we can select a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) with the following properties:
(i) \( \lim_{k} \langle x - Px, J_q(z_{n_k} - Px) \rangle = K \),
(ii) \( \{z_{n_k}\} \) converges to a point \( z \);
by Step 3, we have \( z \in \text{Fix}(T) \). From Theorem 4.4 and since \( J_q \) is continuous, we have
\[
K = \lim_{j} \langle x - Px, J_q(z_{n_k} - Px) \rangle = \langle x - Px, J_q(z - Px) \rangle \leq 0.
\]
Since \( fPx = x \), we have \((f - I)Px = x - Px\). Now, from Proposition 4.10 and our assumption we have, for each \( n \in I \),
\[
\epsilon_n(\beta - 1)q(z_n - Px)^2 \\
\geq \left[ \epsilon_n \beta q(z_n - Px) + (1 - \epsilon_n)q(z_n - Px) \right]^2 - q(z_n - Px)^2 \\
\geq \left[ \epsilon_n q(fz_n - f(Px)) + (1 - \epsilon_n)q(Tz_n - Px) \right]^2 - q(z_n - Px)^2 \\
\geq 2\left( \epsilon_n \left( f(z_n - f(Px)) \right) + (1 - \epsilon_n)(Tz_n - Px) - (z_n - Px) \right) \\
= -2\epsilon_n \langle (f - I)Px, J_q(z_n - Px) \rangle \\
= -2\epsilon_n \langle x - Px, J_q(z_n - Px) \rangle,
\]

hence,
\[
q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \langle x - Px, J_q(z_n - Px) \rangle. \tag{4.19}
\]

for each \( q \in Q \).

Step 5. \( \{z_n\} \) converges to \( Px \) in \( \tau_Q \).

Proof. Indeed, from (4.18), (4.19) and that \( Px \in \text{Fix}(T) \), we conclude
\[
\limsup_n q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \limsup_n \langle x - Px, J_q(z_n - Px) \rangle \leq 0,
\]
for each \( q \in Q \). That is \( z_n \to Px \) in \( \tau_Q \). \( \square \)

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REFERENCES


